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MATHEMATICAL THEORY FOR SEISMIC MIGRATION AND SPATIAL RESOLUTION

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INTRODUCTION

Migration methods in seismics are computationally intensive and powerful tools for interpretation of seismic experiments. A partial list of papers on migration methods in the geophysical literature include those by Hagedoorn (1954), Lindsey and Herman (1970), Rockwell (1971), Claerbout (1971), Schneider (1971, 1978), Claerbout and Doherty (1972), French (1974, 1975), Gardner *et al.* (1974), Cohen and Bleistein (1977, 1979), Stolt (1978), Berkhout (1980, 1984), Clayton and Stolt (1981), Johnson and French (1982), Devaney (1984), Gazdag and Squazzero (1984), Tarantola (1984), Bleistein and Gray (1985), Stolt and Weglein (1985) and many more. This list is not intended to be complete and is compiled to show a continuing interest in this subject. Some of the ideas relevant to migration procedures can even be traced back to the nineteen-twenties (see Gardner 1985). Though mathematically involved, migration methods were primarily based on semi-heuristic arguments. More precisely, answers for simple models were routinely extended beyond their limits of applicability by heuristic arguments, either explicitly – in the form of imaging principles, or implicitly – through the choice of weights in Kirchhoff-type migration. This heuristic approach was especially evident for migration with variable background velocity and complex source-receiver configurations.

Recently, the theory of pseudodifferential and Fourier integral operators has been found to provide a mathematically rigorous justification for migration methods. This theory allows the analysis of the most general case of both lateral and vertical variations in the background velocity (and/or other parameters) and arbitrary configurations of sources and receivers. The important role in this approach belongs to the generalized Radon transform (GRT) and its inversion procedure (Gel'fand *et al.* 1969; Guillemin and Sternberg 1977; Quinto 1980; Beylkin 1982, 1984). Miller (1983) recognized that seismic imaging could be cast as the problem of inverting a GRT. Mathematical theory describing the formal derivation of the inversion procedure and the nature of the reconstructed image is described in Beylkin (1985a, 1985b). Geophysical applications as well as the important heuristics of the method are treated in Miller *et al.* (1984, 1987). An example of migration of the field data set can be found in Miller and Dupal (1987), and Dupal and Miller (1985). Though most of the work seems to concentrate around the GRT, the more general nature of the results is apparent (Beylkin 1985b; Beylkin *et al.* 1985; Chang *et al.* 1987): it became clear that the derivation of migration algorithms to solve the linearized inverse scattering problem is best understood within the theory of pseudodifferential and Fourier integral operators.

The sequence for the derivation of a migration algorithm within this theory can be briefly described as follows. For a given background model the inverse problem is linearized using the single scattering (Born) approximation. This is the standard perturbation technique which yields the scattered field as a volume integral of the perturbation. This integral has an oscillatory kernel. Given the scattered field a Fourier integral operator (which can be viewed as being applied to the perturbation of the model parameters) can be constructed. Then the Fourier integral operator is inverted modulo a smooth error. This inversion procedure modulo a smooth error is, in fact, a migration algorithm. This connection can be explicitly established by direct comparison with known migration algorithms. For example, considering the generalized Radon transform one can relate inversion of the Fourier integral operator to the Kirchhoff-type migration. The smooth error term can be small in a number of important cases. In all cases the locations of the discontinuities and the sizes of the jumps of the parameters are recovered.

This approach also explains why heuristic arguments worked so well in seismics. Since such arguments led to correct phase predictions (correct travel times), the locations of discontinuities (in the velocity profile, for example) were recovered correctly. Meanwhile, inaccuracies in measurements of amplitudes and preprocessing made amplitude information less useful for recovery of the exact jumps at these discontinuities. As a result, inconsistencies in heuristic arguments with respect to the treatment of the amplitude information were relatively unimportant in practice. However, given accurate amplitudes it appears possible to recover the sizes of the jumps at discontinuities correctly. This, in turn, gives an objective criterion for comparing different implementations of migration algorithms. In fact, this approach provides a unified point of view on Kirchhoff-type migration and 'full wave equation' migration (Beylkin *et al.* 1985; Levy and Esmersoy 1987).

A major advantage gained by embedding the analysis into the theory of pseudodifferential operators – apart from mathematical consistency – is the explicit description of the spatial resolution of migration algorithms and its dependence on limited apertures of seismic experiments and limited frequency content of seismic sources (Beylkin *et al.* 1985). Analysis of the spatial Fourier spectrum of the object yields a local version of relationships described by Wolf (1969) in the context of holographic imaging. We note that the relationship of the spatial Fourier spectrum of the object to the data is also the basis of diffraction tomography (Devaney 1982, 1984; Devaney and Beylkin 1984) and is used implicitly in migration by Fourier transform (Stolt 1978). The conclusion (Beylkin *et al.* 1985) is that the theoretical limit of spatial resolution does not depend on the type of migration algorithm used in reconstruction, but does depend on the background model, on the configuration of sources and receivers, and on the frequency content of the source.

This presentation is intended to demonstrate in greater detail connections of the theory of pseudodifferential and Fourier integral operators with the derivation of migration procedures and estimation of spatial resolution of these algorithms.

17.1 Pseudodifferential and Fourier Integral Operators

The title of this section might discourage a non-mathematician since these notions are not yet a part of the standard curriculum in applied mathematics. However, these mathematical objects are well known under different names to electrical engineers and to those who process images. In these fields, pseudodifferential operators are known as space- or time-varying filters. A specific example of a two-dimensional pseudodifferential operator would be a spatially varying edge detection operator in image processing.

It is natural to ask: What does the mathematical theory of pseudodifferential and Fourier integral operators contribute to applications? I will try to answer this question with respect to migration (or inversion) algorithms. I will be emphasizing two points which are standard in this theory, namely, the classification of pseudodifferential operators and the machinery of formal manipulation with these operators. The purpose of this section is to give a very brief introduction to the theory of pseudodifferential operators in preparation for treatment of migration algorithms. As we will see in what follows, routine application of very basic ideas developed in this theory will yield migration schemes suitable for arbitrary source-receiver geometries and arbitrary background (or reference) velocities, and also will produce very simple and practical estimates of spatial resolution. And these considerations have sufficiently general direct practical implications.

Classes of Pseudodifferential Operators

We start with the definition of the simplest (but quite sufficient for our purposes) classes of pseudodifferential operators. Initially, we will consider these operators in spaces of arbitrary dimension d.

Consider the operator $\mathbf{P}$,

$$(\mathbf{P}f)(x)=\frac{1}{(2\pi)^d}\int_{R^d} p(x,k)\hat{f}(k)e^{ik\cdot x}\,dk, \tag{17.1}$$

where $\hat{f}$ denotes the Fourier transform of the function f,

$$\hat{f}(k)=\int_X f(y)e^{-ik\cdot y}\,dy. \tag{17.2}$$

The function $p(x, k)$ is called the symbol of the pseudodifferential operator $\mathbf{P}$.

Example. Consider a partial differential operator with variable coefficients

$$\mathbf{P}=\sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha, \tag{17.3}$$

where α is a multiindex, $\alpha=(\alpha_1,\ldots,\alpha_d)$, $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_d$, $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\ldots\partial_{x_d}^{\alpha_d}$, and $\partial_{x_j}=\frac{1}{i}\frac{\partial}{\partial x_j}$. The symbol of this operator is

$$p(x,k)=\sum_{|\alpha|\le m} a_\alpha(x)k^\alpha. \tag{17.4}$$

A more symmetric form of the symbol with respect to variables x and y can be obtained if the expression for $\hat{f}$ in (17.2) is substituted in (17.1) and the symbol is allowed to depend on variable y,

$$(\mathbf{P}f)(x)=\frac{1}{(2\pi)^d}\int_{R^d}\int_X p(x, y, k)f(y)e^{ik\cdot(x-y)}\,\mathrm{d}y\mathrm{d}k. \tag{17.5}$$

Definition. Let X be an open subset of R^d and m be a real number. Let $S^m(X\times X)$ be the class of symbols consisting of infinitely differentiable functions $p(x, y, k)$, $p(x, y, k)\in C^\infty(X\times X\times R^n)$, such that to every compact $Q\subset X$ and to every three multiindices α, β, γ there is a constant $C_Q(\alpha, \beta, \gamma)$, such that

$$|\partial_k^\alpha\partial_x^\beta\partial_y^\gamma p(x, y, k)|\leq C_Q(\alpha, \beta, \gamma)(1+|k|)^{m-|\alpha|}, \tag{17.6}$$

for $x, y\in Q$.

The pseudodifferential operator $\mathbf{P}$ is said to belong to the class $L^m(X)$ if its symbol $p(x, y, k)$ belongs to $S^m(X\times X)$.

This definition of classes $L^m(X)$ grew out from an observation that the operator of differentiation ∂_x^α in the Fourier domain (the operator of multiplication by $k^\alpha=k_1^{\alpha_1}k_2^{\alpha_2}\ldots k_d^{\alpha_d}$) is a homogeneous function of degree $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_d$. Derivatives of homogeneous functions are also homogeneous functions with degree which is less than the original degree by the number of derivatives. For example

$$\partial_{k_j}k^\alpha=\alpha_j k_1^{\alpha_1}k_2^{\alpha_2}\ldots k_j^{\alpha_j-1}\ldots k_d^{\alpha_d}. \tag{17.7}$$

is a homogeneous function of degree $|\alpha|-1$. The definition of classes of symbols depicts this property with respect to differentiation in a weak form through the inequality (17.6), which is useful for estimates. This definition says, roughly, that symbols from the class S^m define operators that are somewhat similar to the operator that takes m derivatives. (We note, however, that m may be any real number). Alternatively, symbols from the class S^{-m} define operators that are somewhat similar to operators that take m integrals.

In more precise terms here are some of the principal properties describing $\mathbf{P}$ as an operator. If $p(x, y, k)\in S^m(X\times X)$ then $\mathbf{P}$ is a continuous operator

$$\mathbf{P}: C_0^\infty(X)\rightarrow C^\infty(X), \tag{17.8}$$

where $C_0^\infty(X)$ denotes the class of infinitely differentiable functions with a compact support in X. The operator $\mathbf{P}$ can be extended to a continuous map

$$\mathbf{P}: \mathscr{E}'(X)\rightarrow \mathscr{D}'(X), \tag{17.9}$$

where $\mathscr{D}'(X)$ is the space of distributions on X (the dual of $C_0^\infty(X)$) and $\mathscr{E}'(X)$ is the space of distributions with compact support (the dual of $C^\infty(X)$).

Theorem. Let $\mathbf{P}$ be a pseudodifferential operator in X of class $L^m(X)$. Given any real number s the operator $\mathbf{P}$ can be extended as a continuous map

$$\mathbf{P}: H^s_{\mathrm{comp}}(X)\rightarrow H^{s-m}_{\mathrm{loc}}(X), \tag{17.10}$$

where $H^s_{\mathrm{comp}}(X)$ and $H^{s-m}_{\mathrm{loc}}(X)$ are the Sobolev spaces of distributions.

The Sobolev space $H^s(R^d)$ is the space of distribution in R^d whose Fourier transform is a square-integrable function in R^d with the measure $(1+|k|^2)^s \, dk$. It is a Hilbert space with the inner product

$$(u, v)=\frac{1}{(2\pi)^d}\int_{R^d}\hat{u}(k)\bar{\hat{v}}(k)(1+|k|^2)^s \, dk, \tag{17.11}$$

where the bar denotes complex conjugation.

The subspace $H^s_{\text{comp}}(Q)$ of $H^s(R^d)$ consists of the distributions with the support in the compact set Q; $H^s_{\text{comp}}(X)$ is the union of the spaces $H^s_{\text{comp}}(Q)$, where Q spans the collection of all compact subsets of X. Finally, $H^s_{\text{loc}}(X)$ is a space of distributions in X, such that if properly localized (by infinitely differentiable cutoff functions to compact subsets exhausting X) these distributions belong to H^s. The index s can be interpreted as a 'number of derivatives' and the theorem describes how this number is affected by a pseudodifferential operator. Indeed, following this interpretation if a function originally has s derivatives then after application of an operator from the class L^m (which acts like taking m derivatives) the function has $s-m$ derivatives. For detailed descriptions see the references on pseudodifferential operators (e.g. Hormander 1965; Kohn and Nirenberg 1965; Treves 1980; Taylor 1981).

Parametrix and asymptotic expansions.

One of the most important tools in the theory of pseudodifferential operators is the parametrix. The parametrix is a solution of a partial differential or an integral equation which is defined exactly like the Green's function except that an arbitrary smooth function may have been added to the source terms of the equation. The nice thing about a parametrix is that, unlike the Green's function, it can be constructed explicitly even for equations with variable coefficients. (The new element here is the notion of parametrix itself because, as it turns out, the construction of an asymptotic Green's function by the ray method produces a parametrix).

More accurate description of the parametrix requires the following:

Definition. The operator $\mathbf{P}$ is said to be regularizing if it maps

$$\mathbf{P}: \mathscr{E}'(X) \rightarrow C^{\infty}(X). \tag{17.12}$$

Let $L^{-\infty}(X)$ be the intersection of all $L^m(X)$, where m is real. One can prove that every operator from the class $L^{-\infty}(X)$ is regularizing and every regularizing operator can be represented as an operator from the class $L^{-\infty}(X)$. This means that a regularizing operator transforms functions with singularities into infinitely smooth functions. If we construct the inverse of an operator modulo regularizing operators then we have a parametrix. It also can be formulated as follows: if we allow *smooth* errors as opposed to *small* errors, then it is sufficient to construct a parametrix instead of the Green's function.

Closely related to the notion of the parametrix is the idea of asymptotic expansion of pseudodifferential operators with respect to smoothness. The classification of pseudodifferential operators gives a scale of smoothness for these operators. Given an operator $\mathbf{P} \in L^m$ an asymptotic expansion of this operator can be constructed:

$$\mathbf{P} = \mathbf{T}_m + \mathbf{T}_{m-1} + \mathbf{T}_{m-2} + \dots, \tag{17.13}$$

where

$$\mathbf{T}_j \in L^j(X), \tag{17.14}$$

for $j = m, m-1, m-2, \dots$, and

$$(\mathbf{P} - \mathbf{T}_m - \mathbf{T}_{m-1} - \dots - \mathbf{T}_j) \in L^{j-1}(X), \tag{17.15}$$

for $j = m, m-1, m-2, \dots$. Such an expansion is modulo regularizing operators.

Fourier integral operators

An operator of the form

$$(Ff)(x) = \frac{1}{(2\pi)^d} \int_{R^d} \int_X f(y) A(x, y, k) e^{i\Phi(x, y, k)} \, dy dk, \tag{17.16}$$

where $A \in S^m(X \times X)$ is called the Fourier integral operator provided this integral can be given meaning through regularization. With appropriate restrictions on the phase function Φ (see references on Fourier integral operators, e.g. Hormander 1971; Treves 1980) the integral in (17.6) can be regularized. If $\Phi(x, y, k) = (x - y) \cdot k$, then F is a pseudodifferential operator.

17.2 LINEARIZED INVERSE SCATTERING PROBLEM

We assume for the sake of simplicity that the propagation of waves is governed by the Helmholtz equation, so that the medium is described by just one function – the index of refraction (reciprocal of the velocity).

We denote the region of interest in the medium by X and its boundary by ∂X. Let the region X be three-dimensional, though the specific dimension of X is not essential. For definiteness, we consider the case when the position of the source is fixed. We assume the experiment has a limited aperture and denote the part of the boundary where receivers are located by ∂X_r.

We assume that the index of refraction in the region X is of the form

$$n^2(x) = n_0^2(x) + f(x), \tag{17.17}$$

where the background index of refraction n_0 is known and the perturbation f is non-zero only inside the region X.

We assume that the function $u(s, r, t)$ – the scattered field – is given as a function of time t, source position s, and receiver position r. For fixed s and r the function $u(s, r, t)$ is a single seismic trace. We assume that the scattered field is causal $u(s, r, t) = 0$, for $t < 0$, and, therefore, real and imaginary parts of the scattered field in the frequency domain

$$\hat{u}(s, r, \omega) = \int_{-\infty}^{+\infty} u(s, r, t) e^{i\omega t} \, dt \tag{17.18}$$

satisfy dispersion relations.

Given the background index of refraction n_0 (background model), the linearized inverse scattering problem is that of characterization of the perturbation f using observations of the scattered field u on the boundary ∂X of the region X.

If the propagation is governed by the Helmholtz equation, then the scattered field $\hat{u}(s, r, \omega)$ satisfies within the single scattering (or distorted wave Born) approximation the following integral equation

$$\hat{u}(s, r, \omega) = -\omega^2 \int_X G(y, r, \omega) f(y) G(s, y, \omega)\,\mathrm{d}y, \tag{17.19}$$

where G is the Green's function of the background model.

The Green's function G is the solution of the equation

$$(\nabla_y^2 + \omega^2 n_0^2(y)) G(y, r, \omega) = \delta(y - r), \tag{17.20}$$

and, in principle, can be computed given the background model. The incident field $G(s, y, \omega)$ is due to the point source located at the point s.

Remark 1. Implicit in (17.20) is the assumption that the boundary ∂X is not physical (in our case it means that the index of refraction does not have a jump at ∂X). If the boundary is a physical boundary then the definition of the Green's function changes and includes boundary conditions. In what follows it only affects the computation of amplitudes.

Remark 2. Equation (17.19) is obtained via standard perturbation analysis. The range of validity of this approximation was discussed by a number of authors and is not treated here.

We view (17.19) as an equation for the unknown function f. This is an integral equation with an oscillatory kernel. We proceed to solve this equation by constructing a Fourier integral operator (which can be shown to be a pseudodifferential operator) and to compute the first term of its asymptotics.

We start by noting that the simplest integral equation with an oscillatory kernel is the Fourier transform

$$\hat{f}(k) = \int_X f(y) \mathrm{e}^{-\mathrm{i}k \cdot y}\,\mathrm{d}y. \tag{17.21}$$

The solution of this equation is obtained by applying the adjoint operator (the inverse Fourier transform),

$$f(x) = \frac{1}{(2\pi)^3} \int_{R^3} \hat{f}(k) \mathrm{e}^{\mathrm{i}k \cdot x}\,\mathrm{d}k. \tag{17.22}$$

Substituting (17.21) in (17.22) we obtain a pseudodifferential operator, which is the identity operator in this case.

By analogy with (17.22), considering (17.19) as the integral equation with the oscillatory kernel we apply (what can be called) the normalized adjoint operator to the scattered field. Denoting the result by $f_{\mathrm{est}}(x)$ we write

$$f_{\mathrm{est}}(x) = -\frac{1}{(2\pi)^3} \mathrm{Re} \int_0^\infty \int_{\partial X_r} \frac{\bar{G}(x, r, \omega)}{|G(x, r, \omega)|^2} \frac{\bar{G}(s, x, \omega)}{|G(s, x, \omega)|^2} h(s, r, x) \hat{u}(s, r, \omega)\,\mathrm{d}r\,\mathrm{d}\omega, \tag{17.23}$$

where the function $h(s, r, x)$ is yet to be described. Here Re denotes the real part of the expression and is introduced to limit the integration with respect to ω to $[0, \infty]$; dr is the surface measure on ∂X_r.

By analogy with (17.21) and (17.22) we substitute the expression for the scattered field (17.19) in (17.23) and obtain

$$f_{\text{est}}(x)=\frac{1}{(2\pi)^3}\,\text{Re}\int_0^\infty\int_{\partial X_r}\int_X G(y,r,\omega)G(s,y,\omega)$$
$$\times\frac{\bar{G}(x,r,\omega)}{|G(x,r,\omega)|^2}\,\frac{\bar{G}(s,x,\omega)}{|G(s,x,\omega)|^2}\,h(s,r,x)f(y)\,\mathrm{d}y\,\mathrm{d}r\,\omega^2\mathrm{d}\omega. \tag{17.24}$$

Operator (17.24) is a Fourier integral operator applied to the function f. It is possible to prove (under reasonable assumptions) that (for a wide class of functions h) it is, in fact, a pseudodifferential operator of the class $L^0(X)$. Given this, we proceed to construct the first term of the asymptotic expansion of this operator with respect to smoothness. By choosing h properly, we show that the first term in this asymptotic series is the identity operator. This, in turn, establishes the relation between $f(x)$ and $f_{\text{est}}(x)$.

Step 1. At this step we replace the Green's functions by the leading order term of their high frequency asymptotics. The justification is as follows. If the domain of integration in (17.24) with respect to ω is restricted to a finite interval then the result always has derivatives of all orders. Since the asymptotics is modulo infinitely differentiable functions, only the contribution from high frequencies affects it. Therefore, we can replace the Green's functions in (17.24) by their high frequency asymptotics. We will keep only the leading order term since only this term contributes to the most singular term in the asymptotics with respect to smoothness.

Thus, we replace the Green's functions in (17.24) by

$$G(s,x,\omega)=A(s,x)\mathrm{e}^{\mathrm{i}\omega\phi(s,x)}, \tag{17.25a}$$

$$G(x,r,\omega)=A(x,r)\mathrm{e}^{\mathrm{i}\omega\phi(x,r)}, \tag{17.25b}$$

where the phase functions $\phi(s, x)$ and $\phi(x, r)$ satisfy the eikonal equation

$$(\nabla_x\phi)^2=n_0^2, \tag{17.26}$$

and amplitudes $A(s, x)$ and $A(x, r)$ are solutions of the transport equation (with proper initial conditions)

$$A\nabla_x^2\phi+2\nabla_x A\cdot\nabla_x\phi=0, \tag{17.27}$$

along the rays connecting the source s with the point x and the point x with the receiver r, respectively.

We arrive at

$$f_{\text{est}}(x)$$
$$=\frac{1}{(2\pi)^3}\,\text{Re}\int_0^\infty\int_{\partial X_r}\int_X\frac{A(y,r)A(s,y)}{A(x,r)A(s,x)}\,\mathrm{e}^{\mathrm{i}\omega(\Phi(s,r,x)-\Phi(s,r,y))}h(s,r,x)f(y)\,\mathrm{d}y\,\mathrm{d}r\,\omega^2\mathrm{d}\omega, \tag{17.28}$$

where

$$\Phi(s, r, x) = \phi(s, x) + \phi(x, r), \tag{17.29}$$

is the total travel time between the source, the point x, and the receiver.

Step 2. At this step we localize the computation to the neighbourhood of the point of reconstruction x. If $\varepsilon < |x - y|$, where ε is any positive number, then the result of integration in (17.28) (over the part of the domain X described by this condition) is infinitely differentiable and, therefore, will not affect the asymptotics. If $|x - y| < \varepsilon$ we replace the phase of the exponent by the first term of the Taylor series

$$\Phi(s, r, x) - \Phi(s, r, y) = \nabla_x \Phi(s, r, x) \cdot (x - y), \tag{17.30}$$

and 'freeze' the value of the amplitude terms at the point x. By doing this we account for the most singular term in the asymptotic expansion with respect to smoothness. We obtain from (17.13)

$$f_{\text{est}}(x) = \frac{1}{(2\pi)^3} \operatorname{Re} \int_0^\infty \int_{\partial X_r} \int_X e^{i\omega \nabla_x \Phi(s, r, x) \cdot (x - y)} h(s, r, x) f(y) \mathrm{d}y \, \mathrm{d}r \, \omega^2 \mathrm{d}\omega. \tag{17.31}$$

Step 3. At this step we set $\omega^2 h(s, r, x)$ to be the Jacobian of the change of variables from $\omega \in [0, \infty]$ and $r \in \partial X_r$ to $k \in R^3$. We have

$$k = \omega \nabla_x \Phi(s, r, x), \tag{17.32}$$

so that

$$\mathrm{d}k = h(s, r, x) \mathrm{d}r \omega^2 \mathrm{d}\omega. \tag{17.33}$$

The function $h(s, r, x)$ can be computed by ray tracing using the identity (Beylkin 1985a)

$$h(s, r, x) \mathrm{d}r = n_0^3 (1 + \cos \psi(s, r, x)) \mathrm{d}\Omega, \tag{17.34}$$

where

$$\cos \psi(s, r, x) = \frac{\nabla_x \phi(s, x) \cdot \nabla_x \phi(x, r)}{n_0^2(x)}. \tag{17.35}$$

$\psi(s, r, x)$ is the angle between the two rays traced from the source and from the receiver to the point x; $\mathrm{d}\Omega$ is the standard solid angle measure on the unit sphere. Relation (17.34) describes the rate of change at the point x of the direction of the ray connecting point x with the receiver with respect to the receiver position on the boundary ∂X. The function $h(s, r, x)$ should be non-zero at the point of reconstruction x in order for (17.34) to be valid. Physically, it corresponds to the regularity of the wave field in the background medium (see Beylkin 1985a).

Thus, we obtain from (17.31)

$$f_{\text{est}}(x) = \frac{1}{(2\pi)^3} \operatorname{Re} \int_{D_x} \int_X e^{ik \cdot (x - y)} f(y) \mathrm{d}y \mathrm{d}k \tag{17.36}$$

where $D_x \subset R^3$ is the region in the Fourier space. It is natural to call D_x the domain of coverage in the space of spatial frequencies. This domain is determined by the map (17.32) of the *total domain* of integration (which we define as [*signal frequency band*] $\times\, \partial X_r$) into the space of spatial frequencies. This region controls the spatial resolution of migration algorithms.

If $D_x = R^3$, then we have shown by obtaining (17.36) that

$$f_{\text{est}}(x) = f(x) + (\mathbf{T}f)(x), \tag{17.37}$$

where $\mathbf{T} \in L^{-1}(X)$. It means that if the perturbation f has jump discontinuities then we reconstruct both the location and the size of the jump at these discontinuities.

17.3 Migration Algorithms

The considerations of the previous section lead to formula (17.23) as the algorithm for reconstructing the location and size of the jump discontinuities of the index of refraction. To see that this formula is, in fact, related to migration procedures we will rewrite it so that it resembles Kirchhoff migration.

First, let us point out the relation between the singly scattered field and the generalized Radon transform.

Relation to the generalized Radon Transform.

Consider the transform defined by

$$(Rf)(s, r, t) = \int_X f(x)A(s, x)A(x, r)\delta(t - \phi(s, x) - \phi(x, r))\, dx, \quad t > 0, \tag{17.38}$$

$$(Rf)(s, r, t) = 0 \qquad t \leq 0.$$

This is the causal generalized Radon transform. Substituting (17.25a) and (17.25b) in (17.19) and transforming the result into the time domain yields the relation between the main term of the high frequency asymptotics of the singly scattered field and the causal generalized Radon transform,

$$u(s, r, t) = \partial_t^2 (Rf)(s, r, t). \tag{17.39}$$

Inversion of the causal generalized Radon transform modulo smooth errors yields Kirchhoff-type migration algorithms. The inversion in the three-dimensional space amounts to the generalized backprojection. We recall, that the result of the reconstruction via (17.22) is accurate up to a smooth error and accounts for the most singular term. Therefore, given (17.39) the inversion modulo smooth errors of the generalized Radon transform will yield the same most singular term. Hence, we can obtain the generalized backprojection (or Kirchhoff-type migration) from the computations of the previous section.

Kirchhoff-type migration

Substituting (17.25a) and (17.25b) in (17.23) and rewriting the result in the time domain we have

$$f_{\text{est}}(s) = -\frac{1}{8\pi^2}\int_{\partial X_r} \frac{h(s, r, x)}{A(s, x)A(x, r)} u(s, r, t)\Big|_{t=\phi(s, x)+\phi(x, r)} \, dr, \tag{17.40}$$

which is the Kirchhoff-type migration for the general non-uniform background and an arbitrary configuration of receiver positions.

The following interpretation can be given to this formula

For a given point of reconstruction x, we want to check if there is a reflector at that point. To accomplish this, we integrate the scattered field $u(s, r, t)$ along the time-distance curve $t = \phi(s, x) + \phi(x, r)$ dictated by the background model. If there were a reflector at the point x then along this curve the scattered field is affected to the greatest extent. The weight is chosen so that we obtain the jump of the function f at the point x as a result of such an integration.

This is the heuristics of the Kirchhoff-type migration which is extended to an arbitrary background model and source-receiver configuration. The image created by this formula is the image of the perturbation of a specific parameter rather than the image of the field. Moreover, the estimate of the spatial resolution of the formula is available through the analysis of the map (17.32).

Spatial resolution

Formula (17.23) has two integrals and depending on the order in which they are taken it carries different interpretation. This was discussed in Beylkin *et al.* (1985). The algorithm can be interpreted as Kirchhoff type migration or, alternatively, it is within the spirit of Claerbout's 'full wave equation' migration (with propagators quite different from those used by Claerbout). It is important that despite the difference in interpretation, the *total* domain of integration remain the same in both cases.

In our considerations the position of the source was fixed. If the source position is not fixed, then (in addition to integrating over all frequencies and receiver positions) the integral over all source positions along ∂X_s should be added in (17.23). In this case the total domain of integration is [*signal frequency band*] $\times \partial X_r \times \partial X_s$.

The total domain of integration is always a bounded domain because of limited apertures and bandlimited signal. It determines the spatial resolution of migration algorithms. It is transformed under the map in (17.32) into the domain of spatial frequencies to produce the region of coverage D_x in (17.36). The description of D_x is, in fact, the estimate of the spatial resolution since it describes what part of the spatial Fourier spectrum is available.

The spatial resolution at a given point x defined by the region D_x depends on

i) the total domain of integration, which is determined by the configuration of sources and receivers and the frequency band of the signal, and

ii) the map (17.32) of the total domain of integration into the domain of spatial frequencies, which is determined by the background model. In general, this map is different for each point of reconstruction.

Concluding remarks

This paper demonstrates (without proofs) how to derive migration algorithms using tools from the theory of pseudodifferential and Fourier integral operators. The purpose of this presentation is to discuss the mathematical technique as it is applied in the context of seismic problems rather than to propose a specific algorithm. For this reason, instead of giving numerical examples, I refer to the papers Miller *et. al.* (1984, 1987), and Beylkin *et. al.* (1985), where specific algorithms are presented along with numerical examples, and to the papers Miller and Dupal (1987) and Dupal and Miller (1985), where the results of applying the algorithms to the field data are discussed.

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