Mathematical and Computational Methods in Seismic Exploration and Reservoir Modeling

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(where $\sigma = \sigma_R + i\sigma_I$, $\sigma_R \neq 0$ is a complex scalar, x and k are n-dimensional real and complex vectors respectively). For each j, $\frac{\partial u}{\partial k_j}$ can be expressed in terms of u and a quantity T which by analogy to the one-dimension case they call the scattering transform of u. An explicit condition characterizing admissible T is obtained from the compatibility requirements $\frac{\partial^2 u}{\partial k_i \partial k_j} = \frac{\partial^2 u}{\partial k_j \partial k_i}$. A certain carefully chosen limit then leads to a family of solutions u of the time-dependent Schrödinger equation ($\sigma = i$) and a corresponding T. Some of the features of the scattering transform T which make it a very attractive object of study are: a) admissible T can be explicitly characterized; b) given the physical scattering amplitude A, T can be found by solving a simple linear integral equation; c) the potential can be reconstructed from T purely by quadratures. When the potential v is independent of time, the solution u one arrives at by the route indicated above, coincides with the one first introduced by Faddeev; the characterization of A via that of T turns out to be essentially equivalent to his. For this problem, some of the new features brought about by the Nachman-Ablowitz approach are: a) a systematic method for deriving as well as understanding Faddeev's characterization; b) the characterization condition is expressed as an integral equation, possibly easier to verify than Faddeev's analyticity requirement (this integral equation also helps explain why there are no simple analogues of the KdV equation associated with higher-dimensional Schrödinger problems); c) a much simpler reconstruction procedure: once T is computed from ("on-shell" A) by solving the linear integral equation (known to Faddeev and Newton)

$$T(\xi,k,\gamma) = A(\xi,k) + c|k|^{n-2} \int_{\substack{\gamma\cdot\eta>\gamma\cdot k \\ |\eta|=|k|}} T(\eta,k,\gamma)A(\xi,\eta)\,d\eta$$

v can be calculated from T purely by quadratures; d) the realization that the mathematical scattering transform T can be explicitly characterized (without any additional miraculous requirements). The latter fact may turn out to be essential if one is to reasonably invert noisy data which may not satisfy the characterization condition.

4. A Mathematical Theory for Reconstructing Discontinuities in Linearized Inverse Problems of Wave Propagation, G. Beylkin*

We develop an approach based upon the theory of Fourier Integral Operators for reconstructing discontinuities of parameters describing a physical medium in linearized inverse problems. Solutions of linearized inverse scattering problems form a mathematical basis for interpretation of seismic reflection data, ultrasound reflectivity imaging in medical applications, crack and void detection and various other methods of non-destructive evalution.

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To illustrate our approach, we consider a medium where wave propagation is described by the Helmholtz equation. Suppose the index of refraction in some region X is of the form $n^2(x) = n_0^2(x) + f(x)$, where $n_0(x)$ -- the background index of refraction--is known. Then the problem is to characterize the function $f(x)$ using observations of the (singly) scattered field on the boundary ∂X of the region X. The incident field is generated by a point source at the point η located outside the region of interest. Let the region X be three-dimensional, however, the specific dimension of X is not essential in our approach, and enters only as a parameter. We treat the case of a variable background index of refraction and arbitrary configuration of sources and receivers.

The linearized inverse scattering problem is formulated in terms of an integral equation of the first kind with an oscillatory kernel relating the singly scattered field to the perturbation $f(x)$

$$v_{sc}(k,\xi,\eta) = -k^2 \int_X G(k,\xi,x)f(x)G(k,\eta,x)dx \ , \tag{1}$$

where ξ and η denote locations of receiver and source, respectively. The Green's function G is the solution of the equation

$$(\nabla_x^2 + k^2 n_0^2)G(k,\xi,x) = \delta(x - \xi) \ .$$

A system of equations of similar structure can be obtained for fluids with variable densities and for elastic solids.

Many practical problems of non-destructive evaluation can be solved provided we can accurately reconstruct discontinuities--location and jump--of parameters of a physical medium. The method for solving (1) that we develop accomplishes this.

Integral equation (1) is connected with the causal Generalized Radon Transform (GRT) if we use geometric optics approximation for both Green's functions in (1),

$$G(k,\ldots,x) = A(\ldots,x)e^{ik\phi(\ldots,x)} \ , \tag{2}$$

where $\phi(\ldots,x)$ is the phase which satisfies the eikonal equation and $A(\ldots,x)$ is the amplitude which satisfies the transport equation. ... here stands for either ξ or η.

If we define a causal Generalized Radon Transform (GRT) as

$$\begin{aligned} (Rf)(t,\xi,\eta) &= \int f(x)a(x,\xi,\eta)\delta(t - \phi(x,\xi,\eta))dx, && \text{for } t > 0, \\ (Rf)(t,\xi,\eta) &= 0, && \text{for } t \le 0, \end{aligned} \tag{3}$$

where

$$\phi(x,\xi,\eta) = \phi(\xi,x) + \phi(\eta,x) ,$$

and

$$a(x,\xi,\eta) = A(\xi,x)A(\eta,x) ,$$

then within the geometric optics approximation $v_{sc}(k,\xi,\eta) = -k^2(Rf)\hat{}(k,\xi,\eta)$, where $\hat{}$ denotes the one-dimensional Fourier transform of (3) with respect to variable t.

Now the problem of solving (1) can be cast as an inversion problem for GRT. The inversion of the GRT requires the introduction of Fourier Integral Operators (FIO). A special role is played by a FIO of the form $F = R^*KR$. Here, R denotes the GRT, R^* is an operator dual to R, and K is a one-simensional convolution operator. R^* is also known as the Generalized Backprojection Operator (GBO). By properly choosing the convolution operator K and the weight function of the GBO the problem of inverting the GRT is reduced to that of solving a Fredholm integral equation.

Exploiting the fact that F is "almost" the identity operator we rigorously establish a class of migration algorithms as approximate solutions of the linearized inverse scattering problem. We prove that

$$F = I + T_1 + T_2 + \dots ,$$

where $T_1, T_2, \dots$ belong to increasingly smooth classes of pseudo-differential operators, $T_j \in L^{-j}(X)$, for $j = 1,2,\dots$ and $F - I \in L^{-1}(X)$. An operator from the class $L^{-1}(X)$ increases by one the number of derivatives of a function to which it is applied. The approximation amounts to using only the first term of an asymptotic expansion for the solution of the integral equation (1), which in terms of GRT can be written as

$$R^*KR \simeq I.$$

Due to the nature of the asymptotics we give a precise meaning to what is reconstructed by this first order inversion for arbitrary configurations of sources and receivers, including the case of limited view angles. In particular, we show that the location of discontinuities and the jump at such discontinuities of the unknown function f are recovered.

We also note that since we reconstruct only discontinuities of the function f, the kernel in (1) can be changed without affecting the first order inversion as long as high frequency asymptotics of it remains the same. Therefore, the use of geometric optics approximation for Green's function (2) does not affect the answer obtained.

Our method yields an algorithm for recovering these discontinuities for variable background velocity and an arbitrary configuration of sources and receivers. The derivation is valid as long as certain physically meaningful conditions on the global structure of rays are satisfied.

Algorithms obtained in this manner have already been used for computations and proved to be robust. They have a simple physical interpretation and for a constant background velocity and specific source-receiver geometries, are directly related to what are known as migration algorithms in seismic exploration.

Our method also allow partial reconstructions for a limited aperture and, in this case, limits on spatial resolution are explicitly obtained.

5. A Marchenko-Faddeev Method for Inverse Scattering in R^3; Roger Newton*

Roger Newton reported on two new inverse scattering methods for the Schrödinger equation. These methods are simple and elegant, but both require the unjustified assumption that a key integral equation, namely Faddeev's version of the Lippmann-Schwinger equation, is uniquely solvable at all points on the real axis. In other words, these methods require that there be no exceptional points.

The first method uses Faddeev's Green's function to obtain a Marchenko-like method. The Faddeev Green's function G_γ satisfies $(\Delta + k^2)G_\gamma = \delta$ and depends on a direction γ as well as on other variables. The function G_γ also has certain analyticity properties. This Green's function is used to construct solutions $\phi_\gamma^+(k,x)$ and $\phi_\gamma^-(k,x)$ of the Schrödinger equation analogous to the usual incoming and outgoing scattering solutions. A quantity h which is analogous to the scattering amplitude can be extracted from the large-x asymptotics of ϕ_γ^+ and ϕ_γ^-. This "scattering amplitude" h is then used to construct an integral operator Ω, which satisfies

$$\phi_\gamma^- - \phi_{-\gamma}^+ = \Omega\, \phi_\gamma^+ .$$

This relation, together with analyticity and asymptotic properties of $\phi_\gamma^\pm$, forms a generalized Riemann-Hilbert problem. It can be Fourier transformed to yield a generalized Marchenko equation. The potential

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