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On Factored FIR Approximation of IIR Filters[1]

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We present a simple algorithm for the factored polynomial (Finite Impulse Response, FIR) approximation of rational (Infinite Impulse Response, IIR) filters which may be used to construct inverse filters. When applied to quadrature mirror filters, our approach yields a simple way of generating an efficient FIR filter bank which inherits the properties of IIR filter bank with any desired accuracy. © 1995 Academic Press, Inc.

I. INTRODUCTION

In this paper we describe a simple and accurate method of approximating Infinite Impulse Response (IIR) filters by Finite Impulse Response (FIR) filters. Let $H(z)$ be an IIR filter,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \tag{1.1}$$

such that

$$H(z) = \frac{P(z)}{Q(z)}, \tag{1.2}$$

where

$$P(z) = \sum_{k=0}^{N_P} b_k z^{-k}, \quad Q(z) = \sum_{k=0}^{N_Q} a_k z^{-k}, \quad a_0 = 1. \tag{1.3}$$

We assume that Q does not vanish on the unit circle. If $Q(z) = 1$ then $H(z)$ is a polynomial in $1/z$ and represents a FIR filter. The class of FIR filters is rather narrow, e.g., the inverse of FIR filter is not an FIR filter. On the other hand the class of IIR filters appears naturally and the inverse filter $H_I(z) = Q(z)/P(z)$ is an IIR filter according to our definition (provided P does not vanish on the unit circle).

Recently a significant attention has been devoted to various constructions of multirate filter banks using Quadrature Mirror Filters (QMFs) (see, e.g., [1, 6] and references therein). These filter banks may be used to generate wavelets and wavelet packets and perform decomposition/reconstruction of functions into wavelet and wavelet packet bases. In this case FIR filter banks correspond to compactly supported wavelets.

It is well known that IIR filters are better as a tool for approximation than FIR filters (rational vs polynomial approximations). Moreover, one has a greater flexibility in the design of IIR filters. For example, it is easier to construct the coefficients for IIR multirate filter banks than FIR filter banks (see, e.g., [6]). It is of interest in this case to find FIR approximations of IIR multirate filter banks.

As it is well known, IIR filters may be implemented via recursive algorithms by solving the standard difference equations. Such implementations result in fast and effective algorithms. However, the control of truncation errors is more difficult for recursive than for nonrecursive implementations since truncation errors tend to accumulate. There has been a significant effort to design the so-called structurally passive IIR filters (see [5] and references therein) with low pass-band sensitivity. Such implementations are typically more accurate than the standard FIR approximations of IIR filters. There are situations, however, where one may choose to use an FIR approximation. As an example, consider a non-causal IIR filter where some of the roots of Q are outside the unit disk. In order to have a stable recursive implementation, it is necessary to access the data in the time-reverse order which presents a problem in some applications.

In finding FIR approximations of IIR filters the standard approach consists in using some optimization criterion (e.g. least squares) for a fixed length of FIR filter (see, e.g., [2]). The reason for fixing the degree of approximating polynomial is to ensure efficiency of the resulting filter. In this paper we depart from the traditional approach and do not

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fix the degree of approximating polynomial. Instead, we attain efficiency by considering a factored FIR approximation with a small number factors where implementation of each factor is inexpensive inspite of the fact that the degree of approximating polynomial might be large. As a result, we may achieve an accurate and efficient implementation which may be viewed as an alternative to structurally passive IIR implementations. In the case of non-causal IIR filters our approach allows one to improve accuracy of the output adaptively as more data become available.

We describe a method for arriving at accurate factored FIR approximations to IIR filters based on the formula

$$\frac{1}{1-z} = \prod_{j=0}^{\infty}(1 + z^{2^j}), \quad |z| < 1, \tag{1.4}$$

which is easily proven by induction with respect to n in $\sum_{k=0}^{2^{n+1}-1} z^k = \prod_{j=0}^{n}(1 + z^{2^j})$. Our construction may be used to generate simple and accurate FIR approximations to inverse filters. We also construct approximate FIR QMFs which satisfy quadrature mirror condition with any desired accuracy. These QMFs may be used for the discrete wavelet and wavelet packet transforms and for computing compactly supported approximations of non-compactly supported wavelets.

II. APPROXIMATION OF IIR BY FIR FILTERS

Let H in (1.2) be an IIR filter, where $P(z)$ and $Q(z)$ in (1.3) are polynomials in $1/z$ of degree N_P and N_Q. We factor Q so that $Q(z) = Q^{\text{in}}(z)Q^{\text{out}}(z)$, where all roots of Q^{in} are inside and all roots of Q^{out} are outside the unit disk. We assume that $Q(z)$ does not have roots on the unit circle.

Factoring $Q^{\text{in}}(z)$ and $Q^{\text{out}}(z)$, we have

$$Q^{\text{in}}(z) = \prod_{k=1}^{N_Q^{\text{in}}}\left(1 - \frac{z_k^{\text{in}}}{z}\right), \tag{2.1}$$

where roots $|z_k^{\text{in}}| < 1, k = 1, \ldots, N_Q^{\text{in}}$, and

$$Q^{\text{out}}(z) = \prod_{k=1}^{N_Q^{\text{out}}}\left(-\frac{z_k^{\text{out}}}{z}\right)\prod_{k=1}^{N_Q^{\text{out}}}\left(1 - \frac{z}{z_k^{\text{out}}}\right), \tag{2.2}$$

where roots $|z_k^{\text{out}}| > 1$, $k = 1, \ldots, N_Q^{\text{out}}$, and $N_Q^{\text{out}} + N_Q^{\text{in}} = N_Q$.

Using (1.4) to expand (2.1) and (2.2), we arrive at

$$H(z) = P(z)\prod_{k=1}^{N_Q^{\text{out}}}\left(-\frac{z}{z_k^{\text{out}}}\right)\prod_{l=1}^{N_Q^{\text{in}}}\prod_{m=1}^{N_Q^{\text{out}}}\prod_{j=0}^{\infty} \times\left[1 + \left(\frac{z_l^{\text{in}}}{z}\right)^{2^j}\right]\left[1 + \left(\frac{z}{z_m^{\text{out}}}\right)^{2^j}\right]. \tag{2.3}$$

Let us formulate conditions for the truncation of the infinite product in (2.3). Given $\epsilon > 0$, for each root inside the unit disk, let $j_l, l = 1, \ldots, N_Q^{\text{in}}$, be such that

$$|z_l^{\text{in}}|^{2^{j_l}} < \epsilon \tag{2.4}$$

and, for each root outside the unit disk, let $j_m, m = 1, \ldots, N_Q^{\text{out}}$, be such that

$$\left|\frac{1}{z_m^{\text{out}}}\right|^{2^{j_m}} < \epsilon. \tag{2.5}$$

We then approximate $H(z)$ by $\tilde{H}(z)$,

$$\tilde{H}(z) = P(z)\prod_{k=1}^{N_Q^{\text{out}}}\left(-\frac{z}{z_k^{\text{out}}}\right)\left(\prod_{l=1}^{N_Q^{\text{in}}}\prod_{j=0}^{j_l}\left[1 + \left(\frac{z_l^{\text{in}}}{z}\right)^{2^j}\right]\right) \times\left(\prod_{m=1}^{N_Q^{\text{out}}}\prod_{j=0}^{j_m}\left[1 + \left(\frac{z}{z_m^{\text{out}}}\right)^{2^j}\right]\right). \tag{2.6}$$

PROPOSITION II.1 *If $j_l, l = 1, \ldots, N_Q^{\text{in}}$ and $j_m, m = 1, \ldots, N_Q^{\text{out}}$ are such that the conditions in* (2.4) *and* (2.5) *are satisfied for some $\epsilon > 0$, then for $|z| = 1$*

$$\frac{|H(z) - \tilde{H}(z)|}{|H(z)|} \leq \epsilon_l + \frac{1}{2}(\epsilon_l)^2\left(1 + \frac{\epsilon_l}{N_Q}\right)^{N_Q-2}, \tag{2.7}$$

where $\epsilon_l = N_Q\epsilon$.

To arrive at (2.7), we note that since

$$\frac{1 - (z_l^{\text{in}}/z)^{2^{j_l+1}}}{1 - (z_l^{\text{in}}/z)} = \prod_{j=0}^{j_l}\left[1 + \left(\frac{z_l^{\text{in}}}{z}\right)^{2^j}\right], \tag{2.8}$$

and

$$\frac{1 - (z/z_m^{\text{out}})^{2^{j_m+1}}}{1 - (z/z_m^{\text{out}})} = \prod_{j=0}^{j_m}\left[1 + \left(\frac{z}{z_m^{\text{out}}}\right)^{2^j}\right], \tag{2.9}$$

we have

$$\tilde{H}(z) = H(z)\prod_{l=1}^{N_Q^{\text{in}}}[1 - (z_l^{\text{in}}/z)^{2^{j_l+1}}]\prod_{m=1}^{N_Q^{\text{out}}}[1 - (z/z_m^{\text{out}})^{2^{j_m+1}}]. \tag{2.10}$$

Therefore,

$$|H(z) - \tilde{H}(z)| = |H(z)||1 - \prod_{l=1}^{N_Q^{in}}[1 - (z_l^{in}/z)^{2^{j_l+1}}] \times \prod_{m=1}^{N_Q^{out}}[1 - (z/z_m^{out})^{2^{j_m+1}}]|. \quad (2.11)$$

For $|z| = 1$ the second factor in (2.11) is of the form $|1 - F(x_1, x_2, \ldots, x_n)|$, where $F(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n}(1 - x_k)$ and $|x_k| < \epsilon, k = 1, \ldots, n$. We have $\partial F/\partial x_k|_{x=0} = -1, k = 1, \ldots, n$, where $x = (x_1, x_2, \ldots, x_n)$ and

$$\left|\frac{\partial^2 F}{\partial x_l \partial x_m}\right| = \left|\prod_{k=1;k\neq l,m}^{n}(1 - x_k)\right| \leq (1 + \epsilon)^{n-2}.$$

Thus, we obtain $|1 - F(x_1, x_2, \ldots, x_n)| \leq n\epsilon + \frac{1}{2}(n\epsilon)^2(1 + \epsilon)^{n-2}$. Applying these considerations to (2.11), we arrive at (2.7).

Remark 1. The approximation of the filter H by $\tilde{H}$ in (2.6) achieves the desired accuracy if the conditions in (2.4) and (2.5) are satisfied. We note that usually the number of factors in (2.6) is not very large. The number of factors slowly increases for the roots close to the unit circle. If z is a root at the distance δ away from the unit circle, then the number of factors for this root may be estimated as $j_l \approx \log_2(-\log_2(\epsilon)) - \log_2(-\log_2(1 - \delta))$.

Remark 2. In order to approximate a non-causal IIR filter, it is necessary to introduce a delay (a number of samples of the input which must be accumulated before a sample of the output can be computed). For non-causal IIR filters ($N_Q^{out} \neq 0$), the delay (the highest degree of z) for the given accuracy ϵ is as follows

$$D_\epsilon = N_Q^{out} + \sum_{m=1}^{N_Q^{out}}\sum_{j=0}^{j_m} 2^{j_m}.$$

In other words, in order to achieve the accuracy ϵ for a non-causal IIR filter, the delay of the output with respect to the input is D_ϵ. Thus, by simply using additional factors, it is possible to improve the accuracy with very little additional computational expense as more and more samples arrive.

EXAMPLE 1. Let us consider the inverse of FIR filter $1 - \alpha z^{-1}$,

$$H(z) = (1 - \alpha z^{-1})^{-1}, \quad (2.12)$$

where $|\alpha| < 1$. Using (2.6), we have

$$\tilde{H}(z) = \prod_{j=0}^{j_l}\left[1 + \left(\frac{\alpha}{z}\right)^{2^j}\right] \quad (2.13)$$

where

$$|\alpha|^{2^{j_l}} < \epsilon. \quad (2.14)$$

In particular, if $\alpha = \frac{1}{2}$, it is sufficient to choose $j_l = 5$ in order to achieve accuracy $\epsilon \approx 2 \cdot 10^{-10}$. If we were to truncate the usual expansion of $H(z)$,

$$H(z) = \sum_{m=0}^{\infty}\frac{\alpha^m}{z^m}, \quad (2.15)$$

then the approximation (2.13) with $j_l = 5$ is equivalent to retaining 64 terms in (2.15). Implementation of the filter in (2.13) requires one multiplication and one addition per factor and, thus, six additions and six multiplications to achieve accuracy $\epsilon \approx 2 \times 10^{-10}$.

If $|\alpha| > 1$, then

$$\tilde{H}(z) = -\frac{z}{\alpha}\prod_{j=0}^{j_l}\left[1 + \left(\frac{z}{\alpha}\right)^{2^j}\right], \quad (2.16)$$

where

$$\left|\frac{1}{\alpha}\right|^{2^{j_l}} < \epsilon. \quad (2.17)$$

In particular, if $\alpha = 2$ and $j_l = 5$, then the truncation error $\epsilon \approx 2 \times 10^{-10}$ and the delay $D_\epsilon = 64$.

III. APPROXIMATION OF IIR QUADRATURE MIRROR FILTERS

The perfect reconstruction filter banks give rise to orthogonal and biorthogonal wavelet bases. Both FIR (see [1]) and IIR (see, e.g., [6]) filters may be found to satisfy the necessary perfect reconstruction condition which we write here for the orthogonal case,

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \quad (3.1)$$

The wavelet bases which correspond to trigonometric polynomial (FIR) and rational IIR solutions of (3.1) have significantly different properties. For example, compactly supported wavelets [1] generated using trigonometric polynomial solutions of (3.1) cannot be symmetric (with the exception of the Haar basis, see, e.g., [1]). On the other hand, rational solutions, or IIR filters, permit the construction of symmetric orthogonal wavelets (see e.g. [6, 3]).

A similar situation exist with respect to the interpolating property of the scaling function. The Fourier transform of the scaling function is defined as

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \qquad (3.2)$$

where $m_0(\xi) = H(e^{i\xi})$. The scaling function has interpolating property if for any f in the subspace spanned by $\{\phi(x-k)\}_{k \in \mathbf{Z}}$,

$$f(x) = \sum_k f(k)\phi(x-k), \qquad (3.3)$$

where $f(k)$ are values of the function f at the integer lattice. Since ϕ also satisfies (3.3) and $\{\phi(x-k)\}$ are orthonormal, we have

$$\begin{aligned}\delta_{m,0} &= \int \phi(x)\phi(x-m)\,dx \\ &= \int \sum_k \phi(k)\phi(x-k)\phi(x-m)\,dx \\ &= \sum_k \phi(k)\delta_{km} = \phi(m). \end{aligned} \qquad (3.4)$$

Thus, on any scale

$$f(x) = \sum_k f(2^{-j}k)\phi(2^j x - k), \qquad (3.5)$$

which is a very convenient property in many applications since values and coefficients are the same. There are no orthogonal compactly supported wavelets with scaling function that has the interpolating property. There exist, however, non-compactly supported wavelets with interpolating scaling function.

In this section we construct compactly supported approximations of non-compactly supported wavelets which over any practical number of scales inherit the properties of non-compactly supported wavelets with any desired accuracy. At the same time these approximate wavelets have properties (e.g., symmetry of interpolating property of the scaling function) which cannot be achieved by their exact counterparts. The approximate quadrature mirror filters may be used to implement the corresponding IIR filters. In such capacity our approach provides an alternative to, for example, [7]. Such implementations are particularly useful for non-causal QMFs in applications where accessing the input data in the time-reverse order presents a problem.

Using (2.10) to approximate $H(z)$,

$$\tilde{H}(z) = H(z)\prod_{l=1}^{N_Q^{\text{in}}}[1-(z_l^{\text{in}}/z)^{2^{j_l+1}}]\prod_{m=1}^{N_Q^{\text{out}}}[1-(z/z_m^{\text{out}})^{2^{j_m+1}}], \qquad (3.6)$$

we have

PROPOSITION III.1. *If $j_l, l = 1, \ldots, N_Q^{\text{in}}$ and $j_m, m = 1, \ldots, N_Q^{\text{out}}$ in (3.6) are such that the conditions in (2.4) and (2.5) are satisfied for some $\epsilon > 0$, then FIR filter $\tilde{H}(z)$ is a solution of*

$$\tilde{H}(z)\tilde{H}(z^{-1}) + \tilde{H}(-z)\tilde{H}(-z^{-1}) = 1 + E(z), \qquad (3.7)$$

where

$$\begin{aligned}E(z) = \prod_{l=1}^{N_Q^{\text{in}}}[1-(z_l^{\text{in}}/z)^{2^{j_l+1}}][1-(z_l^{\text{in}}z)^{2^{j_l+1}}]\prod_{m=1}^{N_Q^{\text{out}}} \\ \times[1-(z/z_m^{\text{out}})^{2^{j_m+1}}][1-(zz_m^{\text{out}})^{2^{j_m+1}}] - 1,\end{aligned} \qquad (3.8)$$

and, on the unit circle $|z| = 1$,

$$|E(z)| \le 2\epsilon\prime + 2(\epsilon\prime)^2\left(1+\frac{\epsilon\prime}{N_Q}\right)^{2N_Q-2} \qquad (3.9)$$

where $\epsilon\prime = N_Q\epsilon$.

Proof of Proposition III.1 is straightforward. We note that on the unit circle $|z| = 1$ $E(z)$ is approximately $2N_Q\epsilon$. By choosing ϵ to be sufficiently small, we find that FIR filter $\tilde{H}(z)$ satisfies the perfect reconstruction condition (3.1) with the error $E(z)$.

EXAMPLE 1 (Butterworth Wavelets). Let us consider IIR QMF filter

$$H(z) = z\frac{(1+z^{-1})^N}{(1+z^{-1})^N + (1-z^{-1})^N} \qquad (3.10)$$

where N is odd. Let us set $N = 3$ for this example so that

$$H(z) = \frac{(1+z)^3}{2(3+z^2)} \qquad (3.11)$$

for the non-causal version of the filter. In order to generate the causal version, z should be replaced by $1/z$. We approximate H by $\tilde{H}$,

$$\tilde{H}(z) = \frac{1}{6}(1+z)^3\prod_{j=0}^{j_1}\left[1+\left(-\frac{z^2}{3}\right)^{2^j}\right]. \qquad (3.12)$$

If we choose $j_1 = 4$, then

$$\begin{aligned}\tilde{H}(z) = \frac{1}{6}(1+z)^3\left(1-\frac{z^2}{3}\right)\left(1+\frac{z^4}{9}\right)\left(1+\frac{z^8}{81}\right) \\ \times\left(1+\frac{z^{16}}{6561}\right)\left(1+\frac{z^{32}}{43046721}\right),\end{aligned} \qquad (3.13)$$

and

$$E_4(z) \approx 2.9\times10^{-31} - 5.4\times10^{-16}\left(z^{64}+\frac{1}{z^{64}}\right). \qquad (3.14)$$

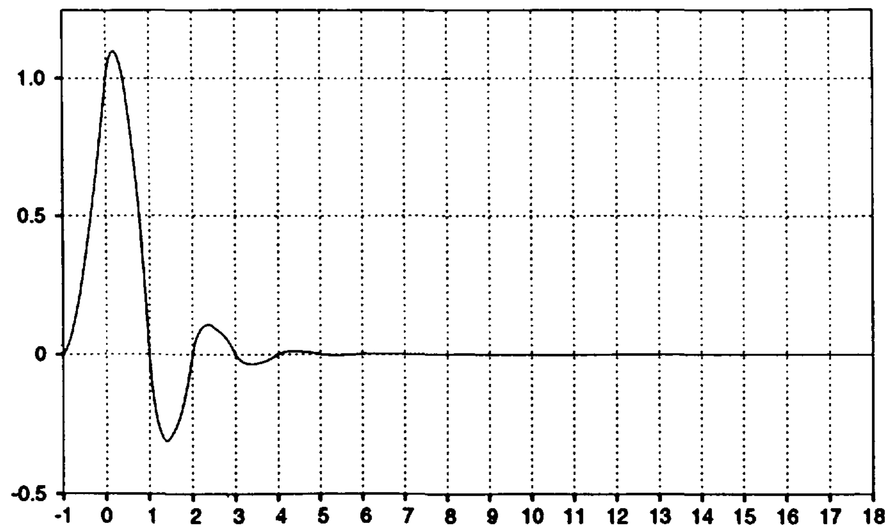

FIG. 1. Compactly supported approximation of Butterworth scaling function $N = 3$ obtained by FIR approximation to QMFs. The support is wider than shown in the picture.

On the unit circle $|E_4| \approx 5.4 \cdot 10^{-16}$. The delay $D = 65$ in this case. If we choose $j_1 = 3$, then

$$E_3(z) \approx 5.4 \cdot 10^{-16} - 2.3 \times 10^{-8} \left(z^{32} + \frac{1}{z^{32}}\right), \quad (3.15)$$

and the delay $D = 33$. Finally, if $j_1 = 5$ then $|E_5| \approx 2.9 \cdot 10^{-31}$. The causal version of the approximate Butterworth filter is obtained by replacing z by $1/z$ in (3.13).

Although Butterworth filters are well-known, it is a recent observation that H satisfies (3.1) and generates noncompactly supported orthonormal wavelets with vanishing moments (see e.g. [6]). It is even more recent observation that these wavelets are interpolating ([4]).

Let us use FIR filter in (3.13) to generate the scaling function for Butterworth wavelets. These compactly supported approximations reproduce the interpolating property of Butterworth wavelets with any given accuracy ϵ. By replacing H by $\tilde{H}$, the error in approximating the scaling function does not exceed ϵ*·number of scales*. FIR generated approximations of Butterworth scaling functions are illustrated in Fig. 1 for $N = 3$ and in Fig. 2 for $N = 5$.

EXAMPLE 2 (Symmetric Wavelets). Let us consider an example of IIR QMFs to generate symmetric wavelets,

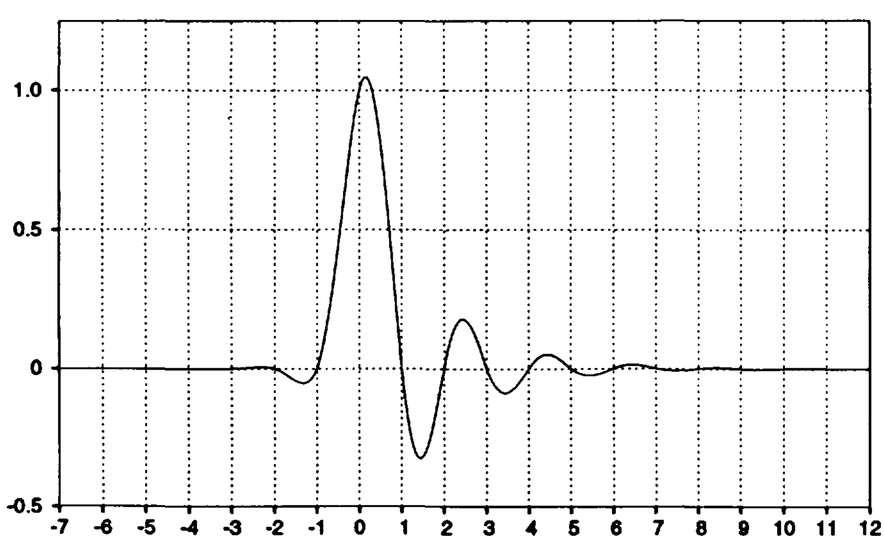

FIG. 2. Compactly supported approximation of Butterworth scaling function $N = 5$ obtained by FIR approximation to QMFs. The support is wider than shown in the picture.

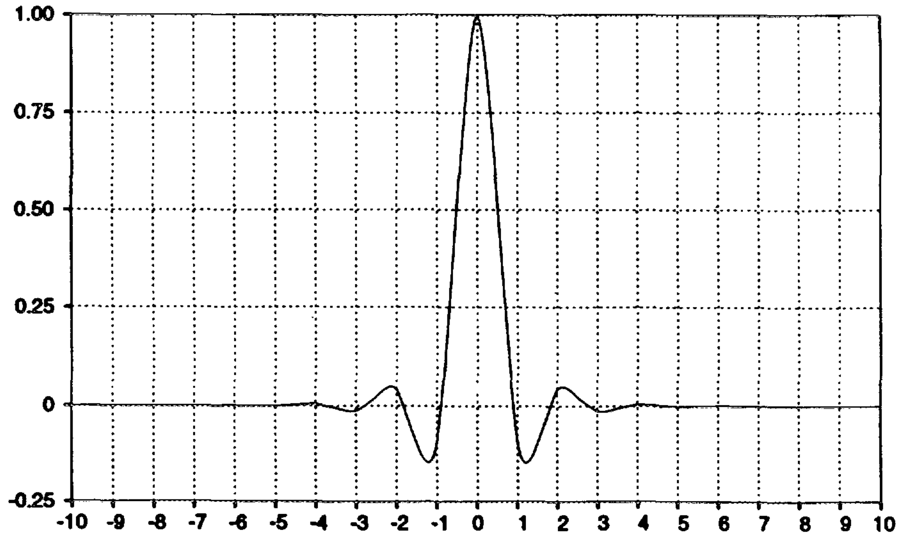

FIG. 3. Compactly supported symmetric scaling function obtained by FIR approximation to QMFs corresponding to symmetric orthonormal wavelets with 4 vanishing moments. The support is wider than shown in the picture but the scaling function is less than 2.229×10^{-6} outside $[-10, 10]$ and less than 1.5×10^{-14} outside $[-25, 25]$.

$$H(z) = \frac{(1+z)^4(\sqrt{2}(1-z)^4 + (1+z)^4)}{(1-z)^8 + (1+z)^8 + \sqrt{2}(1-z^2)^4}. \qquad (3.16)$$

A complete characterization of IIR QMFs for symmetric wavelets (which includes this example) may be found in [3]. The denominator in (3.16) may be factored as

$$(1-z)^8 + (1+z)^8 + \sqrt{2}(1-z^2)^4$$
$$= (z^2 + r_1^2)(z^2 + r_2^2)\left(z^2 + \frac{1}{r_1^2}\right)\left(z^2 + \frac{1}{r_2^2}\right), \qquad (3.17)$$

where $r_1 = \tan^2(3\pi/32)$ and $r_2 = \tan^2(5\pi/32)$. Using Proposition III.1, we construct FIR approximation to the quadrature mirror filters and use these approximate filters to generate the scaling function. The resulting symmetric wavelet where $\epsilon = 10^{-15}$ is illustrated in Fig. 3.

Approximate filters as in (2.6) and (3.13) are implemented as cascaded convolutions. Typically, each root of Q requires 5–6 factors. Thus, factored FIR implementations of IIR filters require about 5–6 times more operations than implementations of IIR filters via recursive algorithms. However, there are several advantages of cascaded implementations that might offset the apparent increase in the number of operations. First, in hardware implementations there is no need to control truncation errors which tend to accumulate in recursive algorithms. Second, for the non-causal filters there is a simple mechanism for improving accuracy of the output. Namely, as the delay increases and more samples of the input signal become available, we may apply additional factors of the approximate filter. Each factor requires one addition and one multiplication but improves the accuracy (due to the given root) quadratically. We refer to Example 2 for the trade-off between the accuracy and the delay.

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