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A Multiresolution Strategy for Numerical Homogenization

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Homogenization may be defined as an analysis in which we construct equations describing coarse-scale behavior of the solution while ignoring fine-scale detail. This work concerns a multiresolution strategy for homogenization of differential equations. As the first step towards a more general treatment of nonlinear ODEs and PDEs, we consider the homogenization via multiresolution analysis (MRA) of systems of linear ODEs with variable coefficients and forcing terms. We develop an efficient numerical approach which generates the coefficients of the homogenized equation. As one of the examples we treat wave propagation in a stratified medium. © 1995 Academic Press, Inc.

1. INTRODUCTION

The term homogenization refers to a collection of methods for the description of the relations between the equations of "microstructure" and those of "macrostructure." It is a diverse field since there is usually more than one way to formulate the problem. We refer to [3, 8] and references therein for examples of various formulations and solutions of problems of homogenization.

Ordinarily, one considers at most two "scales" of variation of the coefficients of the equations governing the microscopic behavior; the goal is to extract the quantities describing the behavior at a coarse scale (maybe as a limit). Thus, the behaviour at possible intermediate scales has been ignored basically due to the lack of tools for its description.

Recently the notion of Multiresolution Analysis (MRA) was introduced by Meyer [9] and Mallat [6] as a general framework for construction of the wavelet bases. Using MRA, the notion of the non-standard representation of operators was introduced in [1]. For a wide class of operators (e.g. Calderón–Zygmund or pseudo-differential operators), the non-standard form is sparse and permits fast algorithms for evaluation of these operators on functions. The non-standard form also permits an explicit description of the interaction between the scales and, thus, appear to be an appropriate tool for the problems of homogenization.

This paper is the first of a series where we use MRA to develop a multiresolution strategy for the numerical solution and homogenization of equations. We consider linear systems of integral equations in one variable, including those equivalent to ODEs and semi-discrete versions of PDEs. In other papers of this series, we plan to consider nonlinear integral equations and equations in more than one variable.

The linear homogenization procedure is exact in that it yields a linear system of equations whose solutions are projections on the coarse scale of the solutions of the original system of ODEs. Moreover, the intermediate systems may be used to describe the behavior of solutions on corresponding scales thus providing a complete description of the transition from fine to coarse-scale representation.

Although our approach may be used with any MRA, we consider separately the case of the Haar basis. To illustrate our approach, let us consider a linear algebraic system

$$Kx = b \tag{1.1}$$

where matrix K is of the size $2^n \times 2^n$. The discrete Haar transform of the vector x is an orthogonal change of basis given by

$$s_k = \frac{1}{\sqrt{2}}(x_{2k-1} + x_{2k}), \quad d_k = \frac{1}{\sqrt{2}}(x_{2k-1} - x_{2k}), \tag{1.2}$$

where $k = 1, \ldots, 2^{n-1}$. Elements of s are scaled averages of neighboring entries, while elements of d are differences. The discrete Haar transform may be written as

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$$M_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & & & \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & \\ \vdots & & & \ddots & & & & \\ 1 & -1 & 0 & 0 & \cdots & & & \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & \\ \vdots & & & \ddots & & & & \end{pmatrix}, \quad (1.3)$$

where matrix M_n is of size $2^n \times 2^n$ and

$$M_n^T M_n = M_n M_n^T = I. \quad (1.4)$$

Let us denote the top half of M_n by L_n and the bottom half by H_n. In this notation, we have by orthogonality

$$M_n^T M_n = H_n^T H_n + L_n^T L_n = I \quad (1.5)$$

and

$$H_n H_n^T = I, \quad L_n L_n^T = I. \quad (1.6)$$

We also note that

$$L_n x = s \quad \text{and} \quad H_n x = d. \quad (1.7)$$

Action of L_n may be thought of as a low-pass filter and that of H_n as a high-pass filter since they tend to separate the low and high frequency components.

Let us split (1.1) into a pair of equations in the unknowns s and d as follows. Applying L to both sides of (1.1) gives

$$LKx = (LKL^T)Lx + (LKH^T)Hx = Lb, \quad (1.8)$$

where we drop subscripts on L_n and H_n. Similarly, we get

$$HKx = (HKL^T)Lx + (HKH^T)Hx = Hb. \quad (1.9)$$

Denoting

$$LKL^T = T, \quad LKH^T = C \quad (1.10)$$

$$HKL^T = B, \quad HKH^T = A, \quad (1.11)$$

$$Lb = b_s, \quad Hb = b_d, \quad (1.12)$$

we have from (1.8), (1.9)

$$Ts + Cd = b_s, \quad (1.13)$$

and

$$Bs + Ad = b_d. \quad (1.14)$$

Let us assume that A is invertible. In this case we may eliminate the unknown d from Eqs. (1.13) and (1.14) to get a reduced system of equations for the unknown s. We have from (1.14)

$$d = -A^{-1}Bs + A^{-1}b_d, \quad (1.15)$$

and, substituting (1.15) into (1.13), obtain

$$(T - CA^{-1}B)s = b_s - CA^{-1}b_d. \quad (1.16)$$

This equation determines s exactly but the number of unknowns has been reduced by half. We will call this the reduction step.

Introducing indices by letting $K_0 = K, b_0 = b$ and

$$K_1 = T_0 - C_0 A_0^{-1} B_0, \quad b_1 = Lb_0 - C_0 A_0^{-1} H b_0, \quad (1.17)$$

we arrive at equation

$$K_1 s_1 = b_1 \quad (1.18)$$

of form similar to (1.1) in the unknown $s_1 = Lx$. The process can be repeated up to n times according to the recursion

$$\begin{aligned} K_{j+1} &= T_j - C_j A_j^{-1} B_j, \\ b_{j+1} &= L_{n-j} b_j - C_j A_j^{-1} H_{n-j} b_j, \end{aligned} \quad (1.19)$$

where

$$T_j = L_{n-j} K_j L_{n-j}^T, \quad C_j = L_{n-j} K_j H_{n-j}^T, \quad (1.20)$$

$$B_j = H_{n-j} K_j L_{n-j}^T, \quad A_j = H_{n-j} K_j H_{n-j}^T. \quad (1.21)$$

If (1.19) is applied a total of n times, then the resulting equation has only a scalar unknown, and the equation is easily solved. The solution of the original equation may then be obtained by a sequence of steps, each step consisting of explicit evaluation of (1.15) followed by reconstruction according to

$$x_{2k-1} = \frac{1}{\sqrt{2}}(s_k + d_k), \quad x_{2k} = \frac{1}{\sqrt{2}}(s_k - d_k). \quad (1.22)$$

Our approach is superficially similar to the elimination of unknowns employed to reduce the size of the system in methods like block Gaussian elimination and cyclic reduction [5]. The difference, however, is that our procedure requires a change of basis that is performed before each reduction step. Thus, the unknowns in the reduced system are not simply a subset of those in the original system. Moreover, we will show in this paper that the vanishing-moments property of the Haar transform allows us to maintain the compressibility of operators at all steps of the procedure. Indeed, it is this property that is necessary for the fast implementation of this procedure, following the methods of [1].

Given the fact that there are many different points of view on homogenization, let us give a simple example to illustrate a numerical approach to homogenization. Let us consider two different equations

$$K_0^1 x^1 = b_0^1, \tag{1.23}$$

$$K_0^2 x^2 = b_0^2, \tag{1.24}$$

such that after one reduction step we obtain $K_1^1, K_1^2, b_1^1, b_1^2$, where

$$K_1^1 = K_1^2 \quad \text{and} \quad b_1^1 = b_1^2. \tag{1.25}$$

Then solutions s^1 and s^2 are identical, i.e.,

$$L_n x^1 = L_n x^2. \tag{1.26}$$

Therefore the solutions of (1.23), (1.24) are identical on the coarse scale, and differ only on the fine scale. This leads us to the basic notion of homogenization. Supposing that one of the systems, say K_0^2, has a more desirable structure than the other, we can then replace (1.23) with (1.24) and be guaranteed that the coarse-scale component of the solution will not change. We then say that (1.24) is the result of homogenization of (1.23).

The outline of this paper is as follows. In Section II we describe a solution strategy for linear equations in a general multiresolution setting, and we state and prove a proposition on sufficient conditions for convergence of this strategy. In Section III, we apply the multiresolution strategy developed in Section II to linear integral equations in one variable using the MRA associated with the Haar basis. Inherent in the multiresolution strategy is a homogenization procedure. In Section IV we consider a homogenization problem associated with linear integral equations in one variable and provide a scheme for its numerical implementation.

In Section V we present numerical examples. The first example is a constant-coefficient scalar wave equation modelling a right-travelling wave. Applying our multiresolution method, we verify the accuracy of the scheme and the elimination of numerical dispersive effects. The second example is a variable-coefficient scalar wave equation which we analyze to show the development of the homogenization procedure through the intermediate scales. Finally we calculate dispersion relations for wave propagation in one-dimensional inhomogeneous media and compare them to the asymptotic result of effective medium theory.

2. MULTIRESOLUTION STRATEGY

In this section we describe the multiresolution strategy in a general setting and defer a specific example where we use the Haar basis to the next section. We would like to work with systems of ordinary differential equations and for this reason we need matrix-valued functions in our considerations. Thus, to formulate our results, we consider functions with values in a Hilbert space.

Let us consider a Hilbert space $\mathcal{H}$ and the equation

$$\mathbf{B}x + q + \lambda = \mathbf{K}(\mathbf{A}x + p), \tag{2.1}$$

where $\mathbf{A}, \mathbf{B}$, and $\mathbf{K}$ are operators on functions in $\mathbf{L}_2(0,1)$ with values in $\mathcal{H}$, λ is a parameter in $\mathcal{H}$, and square-integrable functions q, p, x are defined on $[0, 1]$ with values in $\mathcal{H}$. A representative example is the integral equation

$$(I + B(t))x(t) + q(t) + \lambda = \int_0^t (A(s)x(s) + p(s))\,ds, \quad t \in (0,1), \tag{2.2}$$

where A, B are bounded matrix-valued functions, q, p are vector-valued functions with elements in $\mathbf{L}_2(0,1)$ and λ is a real or complex vector parameter. To put (2.2) into the form (2.1), let $\mathbf{A}$ and $\mathbf{B}$ be the operators whose action is pointwise (matrix) multiplication by A and B, and let $\mathbf{K}$ be the integral operator whose kernel is

$$K(s,t) = \begin{cases} 1 & \text{if } 0 < s < t, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

While the detailed nature of the operators will be important in implementations, the multiresolution strategy can be developed in a completely general context. We use a MRA of $\mathbf{L}_2(0,1)$, i.e., the decomposition of the Hilbert space $\mathbf{L}_2(0,1)$ into a chain of closed subspaces

$$\mathbf{V}_0 \subset \mathbf{V}_{-1} \subset \cdots \subset \mathbf{V}_n \subset \cdots, \tag{2.4}$$

such that

$$\overline{\bigcup_{n \le 0} \mathbf{V}_n} = \mathbf{L}_2(0,1). \tag{2.5}$$

By defining $\mathbf{W}_n$ as an orthogonal complement of $\mathbf{V}_n$ in $\mathbf{V}_{n-1}$,

$$\mathbf{V}_{n-1} = \mathbf{V}_n \oplus \mathbf{W}_n, \tag{2.6}$$

the space $\mathbf{L}_2(0,1)$ is represented as a direct sum

$$\mathbf{L}_2(0,1) = \mathbf{V}_0 \bigoplus_{n \le 0} \mathbf{W}_n. \tag{2.7}$$

Let $\mathbf{P}_n$ and $\mathbf{Q}_n$ be the orthogonal projection operators onto the spaces $\mathbf{V}_n$ and $\mathbf{W}_n$, respectively. We now discretize the integral equation (2.1) by applying the projection operator $\mathbf{P}_n$ with $n \le 0$ to all operators in the equation and look for a solution $x^{(n)}$ in $\mathbf{V}_n$. We find

$$\mathbf{B}_n x^{(n)} + q_n + \lambda = \mathbf{K}_n(\mathbf{A}_n x^{(n)} + p_n), \tag{2.8}$$

where

$$\mathbf{K}_n = \mathbf{P}_n \mathbf{K} \mathbf{P}_n, \tag{2.9}$$

and

$$\mathbf{A}_n = \mathbf{P}_n \mathbf{A} \mathbf{P}_n, \quad \mathbf{B}_n = \mathbf{P}_n \mathbf{B} \mathbf{P}_n, p_n = \mathbf{P}_n p, q_n = \mathbf{P}_n q. \tag{2.10}$$

Let us find a recursion to generate finite sequences $\mathbf{A}_j^{(n)}$, $\mathbf{B}_j^{(n)}, q_j^{(n)}, p_j^{(n)}$ for $j = n, \ldots, 0$ such that $x_j^{(n)} = \mathbf{P}_j x^{(n)}$ satisfies

$$\mathbf{B}_j^{(n)} x_j^{(n)} + q_j^{(n)} + \lambda = \mathbf{K}_j(\mathbf{A}_j^{(n)} x_j^{(n)} + p_j^{(n)}). \tag{2.11}$$

Let us use the nonstandard form [1] to represent the operators. We introduce the following notation:

$$\mathbf{P}_j \mathbf{G}_{j-1}^{(n)} \mathbf{P}_j = \mathcal{T}_{\mathbf{G},j}^{(n)}, \tag{2.12}$$

$$\frac{1}{\epsilon_j} \mathbf{P}_j \mathbf{G}_{j-1}^{(n)} \mathbf{Q}_j = \mathcal{C}_{\mathbf{G},j}^{(n)}, \tag{2.13}$$

$$\frac{1}{\epsilon_j} \mathbf{Q}_j \mathbf{G}_{j-1}^{(n)} \mathbf{P}_j = \mathcal{B}_{\mathbf{G},j}^{(n)}, \tag{2.14}$$

$$\mathbf{Q}_j \mathbf{G}_{j-1}^{(n)} \mathbf{Q}_j = \mathcal{A}_{\mathbf{G},j}^{(n)}, \tag{2.15}$$

where $\mathbf{G}$ is either $\mathbf{A}$ or $\mathbf{B}$ and ϵ_j is a scaling factor that we will choose appropriately for a given MRA. In general, ϵ_j will represent a typical order of magnitude of $\mathbf{Q}_j x^{(n)}$. Also let

$$\mathbf{P}_j \mathbf{K} \mathbf{P}_j = \mathcal{T}_{\mathbf{K},j}, \tag{2.16}$$

$$\frac{1}{\epsilon_j} \mathbf{P}_j \mathbf{K} \mathbf{Q}_j = \mathcal{C}_{\mathbf{K},j}, \tag{2.17}$$

$$\frac{1}{\epsilon_j} \mathbf{Q}_j \mathbf{K} \mathbf{P}_j = \mathcal{B}_{\mathbf{K},j}, \tag{2.18}$$

$$\frac{1}{\epsilon_j^2} \mathbf{Q}_j \mathbf{K} \mathbf{Q}_j = \mathcal{A}_{\mathbf{K},j}. \tag{2.19}$$

PROPOSITION II.1 *The recursion relations for $\mathbf{A}_j^{(n)}$, $\mathbf{B}_j^{(n)}, q_j^{(n)}, p_j^{(n)}$ in terms of $\mathbf{A}_{j-1}^{(n)}, \mathbf{B}_{j-1}^{(n)}, q_{j-1}^{(n)}, p_{j-1}^{(n)}$ are*

$$\begin{aligned}
\mathbf{A}_j^{(n)} &= \mathcal{T}_{\mathbf{A},j}^{(n)} - \epsilon_j^2 \mathcal{C}_{\mathbf{A},j}^{(n)} \mathbf{F}_j^{(n)^{-1}} (\mathcal{B}_{\mathbf{B},j}^{(n)} - \mathcal{B}_{\mathbf{K},j} \mathcal{T}_{\mathbf{A},j}^{(n)} - \epsilon_j^2 \mathcal{A}_{\mathbf{K},j} \mathcal{B}_{\mathbf{A},j}^{(n)}), \\
\mathbf{B}_j^{(n)} &= \mathcal{T}_{\mathbf{B},j}^{(n)} - \epsilon_j^2 \mathcal{C}_{\mathbf{K},j} \mathcal{B}_{\mathbf{A},j}^{(n)} - \epsilon_j^2 (\mathcal{C}_{\mathbf{B},j}^{(n)} - \mathcal{C}_{\mathbf{K},j} \mathcal{A}_{\mathbf{A},j}^{(n)}) \mathbf{F}_j^{(n)^{-1}} \\
&\quad \times (\mathcal{B}_{\mathbf{B},j}^{(n)} - \mathcal{B}_{\mathbf{K},j} \mathcal{T}_{\mathbf{A},j}^{(n)} - \epsilon_j^2 \mathcal{A}_{\mathbf{K},j} \mathcal{B}_{\mathbf{A},j}^{(n)}),
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
q_j^{(n)} &= S_{q,j}^{(n)} - \epsilon_j^2 \mathcal{C}_{\mathbf{K},j} D_{p,j}^{(n)} - \epsilon_j^2 (\mathcal{C}_{\mathbf{B},j}^{(n)} - \mathcal{C}_{\mathbf{K},j} \mathcal{A}_{\mathbf{A},j}^{(n)}) \mathbf{F}_j^{(n)^{-1}} \\
&\quad \times (D_{q,j}^{(n)} - \mathcal{B}_{\mathbf{K},j} S_{p,j}^{(n)} - \epsilon_j^2 \mathcal{A}_{\mathbf{K},j} D_{p,j}^{(n)}), \\
p_j^{(n)} &= S_{p,j}^{(n)} - \epsilon_j^2 \mathcal{C}_{\mathbf{A},j}^{(n)} \mathbf{F}_j^{(n)^{-1}} \\
&\quad \times (D_{q,j}^{(n)} - \mathcal{B}_{\mathbf{K},j} S_{p,j}^{(n)} - \epsilon_j^2 \mathcal{A}_{\mathbf{K},j} D_{p,j}^{(n)}),
\end{aligned} \tag{2.21}$$

$$S_{q,j}^{(n)} = \mathbf{P}_j q_{j-1}^{(n)}, \quad S_{p,j}^{(n)} = \mathbf{P}_j p_{j-1}^{(n)}$$

$$D_{q,j}^{(n)} = \frac{1}{\epsilon_j} \mathbf{Q}_j q_{j-1}^{(n)}, \quad D_{p,j}^{(n)} = \frac{1}{\epsilon_j} \mathbf{Q}_j p_{j-1}^{(n)}.$$

where we assume that the operator

$$\mathbf{F}_j^{(n)} = \mathcal{A}_{\mathbf{B},j}^{(n)} - \epsilon_j^2 \mathcal{B}_{\mathbf{K},j} \mathcal{C}_{\mathbf{A},j}^{(n)} - \epsilon_j^2 \mathcal{A}_{\mathbf{K},j} \mathcal{A}_{\mathbf{A},j}^{(n)} \tag{2.22}$$

is invertible.

Some important special cases where $\mathbf{F}_j^{(n)}$ is not invertible will be discussed in the examples. Proof of Proposition II.1 may be found in Appendix A, and we will refer to some notation introduced in this Appendix.

Recursion relations (2.20) and (2.21) allow us to make a transition from equation

$$\mathbf{B}_{j-1}^{(n)} x_{j-1}^{(n)} + q_{j-1}^{(n)} + \lambda = \mathbf{K}_{j-1}(\mathbf{A}_{j-1}^{(n)} x_{j-1}^{(n)} + p_{j-1}^{(n)}) \tag{2.23}$$

in the subspace $\mathbf{V}_{j-1}$ to equation (2.11) in $\mathbf{V}_j$. Because we have reduced the problem (2.23) to solving (2.11) in $\mathbf{V}_j$, a subspace of $\mathbf{V}_{j-1}$, we call this a reduction step.

The initialization steps for the recursion relations (2.20), (2.21) are

$$\begin{aligned}
\mathbf{A}_n^{(n)} &= \mathbf{A}_n, \quad \mathbf{B}_n^{(n)} = \mathbf{B}_n, \\
p_n^{(n)} &= p_n, \quad q_n^{(n)} = q_n.
\end{aligned} \tag{2.24}$$

The relations (2.20), (2.21) may then be applied sequentially n times to yield

$$\mathbf{B}_0^{(n)} x_0^{(n)} + q_0^{(n)} + \lambda = \mathbf{K}_0(\mathbf{A}_0^{(n)} x_0^{(n)} + p_0^{(n)}), \tag{2.25}$$

an equation in $\mathbf{V}_0$ that reproduces the coarse-scale behavior of the solution $x^{(n)}$ of the discretized system (2.8) for all values of the parameter λ.

Let us now formulate a multiresolution strategy for obtaining the solution $x^{(n)}$. The following are the steps of the procedure:

1. Construct the sequences of operators $\{\mathbf{A}_j^{(n)}\}, \{\mathbf{B}_j^{(n)}\}$, and forcing terms $\{q_j^{(n)}\}, \{p_j^{(n)}\}$ for $n \leq j \leq 0$ using (2.20) and (2.21) (see Figs. 1 and 2);
2. Solve Eq. (2.25) for $x_0^{(n)}$ in the subspace $\mathbf{V}_0$ (see Fig. 3);
3. Obtain the projections of the solutions onto the subspaces $\mathbf{W}_j, j = 0, \ldots, n-1$ by applying (6.16), i.e.,

$$\mathbf{Q}_j x^{(n)} = -\epsilon_j \mathbf{C}_j^{(n)} x_j^{(n)} - \epsilon_j r_j^{(n)} \tag{2.26}$$

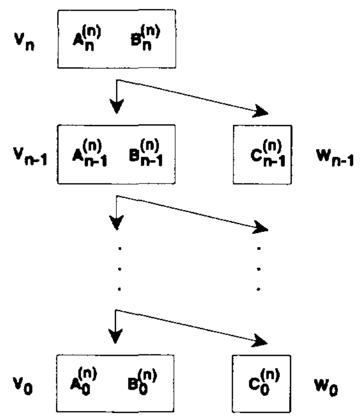

FIG. 1. Schematic diagram of a sequence of reduction steps.

followed by the reconstruction algorithm

$$x_{j-1}^{(n)} = x_j^{(n)} + \mathbf{Q}_j x^{(n)} \tag{2.27}$$

for each j (see Fig. 4).

The first step sweeps through the resolution levels from finest to coarsest, and involves calculations with the operators and forcing terms of the equations. The second step is the solution of a linear system in the coarsest level, $\mathbf{V}_0$. The third step sweeps through the resolution levels from coarsest to finest, determining components of the solution explicitly. Note that the relations (6.17), (2.20) are independent of the forcing terms q, p (see Fig. 1); i.e., it is not necessary to recalculate the sequences (6.17), (2.20) if we change the forcing terms. The relations (6.18), (2.21), which do not involve the forcing terms (see Fig. 2), represent a linear transformation that, when used with the reconstruction steps (2.26), (2.27), effectively produces a multiresolution Green's function.

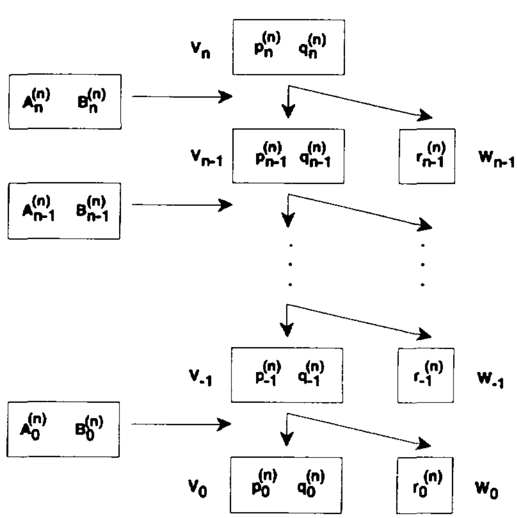

FIG. 2. Schematic diagram of a sequence of reduction steps for forcing terms.

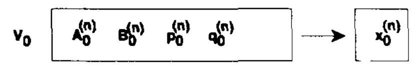

FIG. 3. Schematic diagram of the solution step on the coarsest scale.

Remark. We recall that the parameter λ may be viewed as representing the initial conditions of the differential equation which corresponds to the integral equation (2.2). The results of the first step are independent of the value of the parameter λ. The second and third steps may be carried out for a particular value of parameter λ or, introducing a fundamental solution of the Cauchy problem, in general.

Linear Functionals of the Solution. We may also accumulate other relevant information about the solution during the multiresolution procedure. For example, we may wish to know the value of the solution at the endpoint, or a specific moment of the solution. Since these are linear functionals of the solution, let us outline a multiresolution strategy for evaluating a linear or affine functional of the solution to (2.1).

Let an affine functional $\mathbf{L}$ be defined by

$$\mathbf{L}x = \langle \alpha, x \rangle + \beta. \tag{2.28}$$

Discretization of (2.28) gives

$$\mathbf{L}^{(n)} x^{(n)} = \langle \mathbf{P}_n \alpha, x_n^{(n)} \rangle + \beta. \tag{2.29}$$

We will repeatedly replace $\mathbf{P}_{j-1} x^{(n)}$ with the sum of the projections at the next coarser level of resolution, $\mathbf{P}_j x^{(n)}, \mathbf{Q}_j x^{(n)}$. Furthermore, (2.26) gives us a formula for $\mathbf{Q}_j x^{(n)}$ in terms of $\mathbf{P}_j x^{(n)}$. Thus we find

$$\mathbf{L}^{(n)} x^{(n)} = \langle \alpha_j^{(n)}, x_j^{(n)} \rangle + \beta_j^{(n)}, \tag{2.30}$$

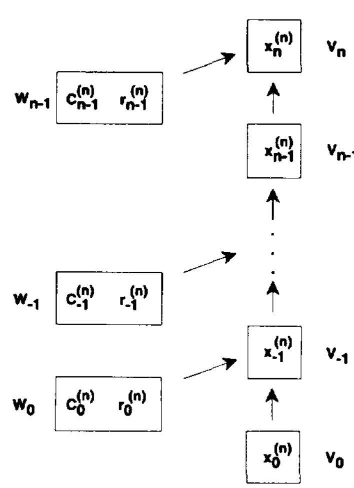

FIG. 4. Schematic diagram of a sequence of reconstruction steps.

where

$$\alpha_n^{(n)} = \mathbf{P}_n\alpha, \quad \beta_n^{(n)} = \beta, \tag{2.31}$$

$$\alpha_j^{(n)} = \mathbf{P}_j\alpha_{j-1}^{(n)} - \epsilon_j\mathbf{C}_j^{(n)}\mathbf{Q}_j\alpha_{j-1}^{(n)} \tag{2.32}$$

and

$$\beta_j^{(n)} = \beta_{j-1}^{(n)} - \epsilon_j\langle \mathbf{Q}_j\alpha_{j-1}^{(n)}, r_j^{(n)}\rangle, \tag{2.33}$$

for $n \leqslant j \leqslant 0$. At the coarsest level we have an explicit formula that provides the value of the functional $\mathbf{L}$ in terms of $x_0^{(n)}$. We may also solve (2.25) to get an explicit formula for the value of the functional in terms of λ

$$\mathbf{L}^{(n)}x^{(n)} = \langle \alpha_0^{(n)}, (\mathbf{B}_0^{(n)} - \mathbf{K}_0\mathbf{A}_0^{(n)})^{-1} \times (\mathbf{K}_0 p_0^{(n)} - q_0^{(n)} - \lambda)\rangle + \beta_0^{(n)}. \tag{2.34}$$

A Limit of the Multiresolution Strategy. Now consider the problem (2.11) in the limit as $n \to -\infty$. We wish to identify sufficient conditions such that in the limit $n \to -\infty$ the solutions $\{x_j^{(n)}\}$ of the equations (2.11) converge to the projections $x_j = \mathbf{P}_j x$ of the solution of the original system.

Let us assume that for fixed j there is a subsequence $\{n_k\}$ such that the limits

$$\begin{aligned} \mathbf{A}_j^{(-\infty)} &= \lim_{k\to-\infty} \mathbf{A}^{(n_k)}, & p_j^{(-\infty)} &= \lim_{k\to-\infty} p^{(n_k)}, \\ \mathbf{B}_j^{(-\infty)} &= \lim_{k\to-\infty} \mathbf{B}_j^{(n_k)}, & q_j^{(-\infty)} &= \lim_{k\to-\infty} q_j^{(n_k)} \end{aligned} \tag{2.35}$$

exist. In our examples we will verify this hypothesis separately. An important example where this condition is *not* met will be presented in Section V.

We now prove the following:

PROPOSITION II.2 *Let $\mathcal{H}$ be a Hilbert space and $\mathbf{L}_2^{\mathcal{H}}$ denote the space of square-integrable functions with values in $\mathcal{H}$. Suppose that $\mathbf{A}, \mathbf{B}, \mathbf{K}$ are bounded operators mapping $\mathbf{L}_2^{\mathcal{H}} \to \mathbf{L}_2^{\mathcal{H}}$ and $\mathbf{B} - \mathbf{KA}$ has a bounded inverse. Then Eq.* (2.1) *has a unique solution $x \in \mathbf{L}_2^{\mathcal{H}}$ for each $q, p \in \mathbf{L}_2^{\mathcal{H}}$. Also, there is some $n_0 \leqslant 0$ such that for all integers $n \leqslant n_0, \mathbf{B}_n - \mathbf{K}_n\mathbf{A}_n$ has a bounded inverse and, therefore,* (2.8) *has a unique solution $x^{(n)} \in \mathbf{V}_n$. Furthermore, the sequence $x^{(n)}$ converges to x.*

PROPOSITION II.3 *Let us assume that in addition to the conditions in Proposition II.2 the recurrence relations* (2.20) *and* (2.21) *give rise to sequences $\{\mathbf{A}_j^{(n)}\}, \{\mathbf{B}_j^{(n)}\}, \{q_j^{(n)}\}, \{p_j^{(n)}\}$ that are well-defined for $n \leqslant n_0 \leqslant 0$ and $n \leqslant j \leqslant 0$. Also, let us assume that the limits* (2.35) *exist.*

Then each $x_j^{(-\infty)} = \mathbf{P}_j x$ satisfies

$$\mathbf{B}_j^{(-\infty)}x_j + q_j^{(-\infty)} = \mathbf{K}_j(\mathbf{A}_j^{(-\infty)}x_j + p_j^{(-\infty)}), \tag{2.36}$$

and $\mathbf{B}_j^{(-\infty)} - \mathbf{K}_j\mathbf{A}_j^{(-\infty)}$ has a bounded inverse,

$$\|(\mathbf{B}_j^{(-\infty)} - \mathbf{K}_j\mathbf{A}_j^{(-\infty)})^{-1}\| \leqslant \|(\mathbf{B} - \mathbf{KA})^{-1}\|. \tag{2.37}$$

Proof of Propositions II.2 and II.3 may be found in Appendix B.

We will verify the hypotheses of Propositions 1 and 2 in some particular cases in the following sections.

3. THE HAAR BASIS

We now consider an example where the recurrence relations (2.20) can be developed explicitly. The problem to be solved is the system of integral equations (2.2), which we restate here

$$(I + B(t))x(t) + q(t) + \lambda = \int_0^t (A(s)x(s) + p(s))\,ds, \quad t \in (0,1), \tag{3.1}$$

where A, B are bounded matrix-valued functions, q, p are vector-valued functions with elements in $\mathbf{L}_2(0,1)$ and λ is a real or complex vector parameter. Such problems arise from the consideration of linear systems of ODEs or from the discretization of PDEs by the method of lines.

Representing the unknown vector $x(t)$ of size N_x in the Haar basis, we obtain

$$x(t) = s_{0,0}\phi_{0,0} + \sum_{j=-\infty}^{0} \sum_{k=0}^{N_j-1} d_{j,k}\psi_{j,k}, \tag{3.2}$$

where

$$\phi_{j,k}(t) = \frac{1}{\sqrt{\delta_j}}\phi\left(\frac{t}{\delta_j} - k\right) \tag{3.3}$$

and

$$\psi_{j,k}(t) = \frac{1}{\sqrt{\delta_j}}\psi\left(\frac{t}{\delta_j} - k\right) \tag{3.4}$$

are dilations and translations of the characteristic function

$$\phi_{0,0}(t) = \phi(t) = \begin{cases} 1 & \text{if } t \in [0,1) \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

and the Haar function

$$\psi_{0,0}(t) = \psi(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{1}{2}\right) \\ -1 & \text{if } t \in \left[\frac{1}{2}, 1\right), \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

respectively and $\delta_j = 2^j, N_j = 2^{-j}$.

Since the Haar basis is orthonormal, the coefficients are obtained by forming the appropriate inner products, namely, averages

$$s_{j,k} = \int_{-\infty}^{\infty} \phi_{j,k}(t)x(t)\,dt = \frac{1}{\sqrt{\delta_j}}\int_{k\delta_j}^{(k+1)\delta_j} x(t)\,dt, \tag{3.7}$$

and differences

$$\begin{aligned} d_{j,k} &= \int_{-\infty}^{\infty} \psi_{j,k}(t)x(t)\,dt \\ &= \frac{1}{\sqrt{\delta_j}}\left(\int_{k\delta_j}^{(k+1/2)\delta_j} x(t)\,dt - \int_{(k+1/2)\delta_j}^{(k+1)\delta_j} x(t)\,dt\right). \end{aligned} \tag{3.8}$$

Truncations of the series over scales in (3.2) give projections of x into the subspaces of the multiresolution analysis. Thus, for $n < 0$,

$$\mathbf{P}_n x(t) = s_{0,0}\phi_{0,0} + \sum_{j=n+1}^{0}\sum_{k=0}^{N_j-1} d_{j,k}\psi_{j,k} \in \mathbf{V}_n, \tag{3.9}$$

and

$$\mathbf{Q}_n x(t) = \sum_{k=0}^{N_n-1} d_{n,k}\psi_{n,k} \in \mathbf{W}_n. \tag{3.10}$$

Note that

$$\mathbf{P}_n x(t) = \sum_{k=0}^{N_n-1} s_{n,k}\phi_{n,k} \tag{3.11}$$

is an alternate representation of the projection of x into $\mathbf{V}_n$, as is

$$\mathbf{P}_n x(t) = \sum_{k=0}^{N_m-1} s_{m,k}\phi_{m,k} + \sum_{j=n+1}^{m}\sum_{k=0}^{N_j-1} d_{j,k}\psi_{j,k}, \tag{3.12}$$

where $m \geqslant n+1$.

To solve the integral equation (2.2), we first find the matrix representations for the discretization in (2.8). The matrix representation for the integral operator $\mathbf{K}$ acting from $\mathbf{V}_n$ into itself is

$$\mathbf{K}_n = \delta_n \begin{pmatrix} \frac{1}{2}I & 0 & \cdots & 0 \\ I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ I & \cdots & I & \frac{1}{2}I \end{pmatrix}, \tag{3.13}$$

where I is the identity matrix of size N_x. We denote by $\mathbf{M}(A)$ the operator whose action is pointwise (matrix) multiplication by A. Thus

$$\mathbf{A}_n = \mathbf{M}(A_n) = \operatorname{diag}\{A_{n,0}, A_{n,1}, \ldots, A_{n,N_n-1}\} \tag{3.14}$$

where the matrix-valued function A_n is given by

$$A_n = \sum_{k=0}^{N_n-1} \sqrt{\delta_n}A_{n,k}\phi_{n,k}. \tag{3.15}$$

The matrix representation $\mathbf{B}_n$ is obtained similarly to that of $\mathbf{A}_n$. Thus, we rewrite (2.8) in the Haar basis,

$$\begin{aligned} (I + B_{n,k})x_{n,k} + q_{n,k} + \lambda = \frac{1}{2}\delta_n(A_{n,k}x_{n,k} + p_{n,k}) \\ + \delta_n\sum_{k'=0}^{k-1}(A_{n,k'}x_{n,k'} + p_{n,k'}). \end{aligned} \tag{3.16}$$

In order to analyse the convergence and accuracy of our method, let us consider a finite difference version of the discretization (3.16). By subtracting successive evaluations of (3.16) for k and $k+1$, we find

$$\begin{aligned} (I + B_{n,k+1})x_{n,k+1} - (I + B_{n,k})x_{n,k} + q_{n,k+1} - q_{n,k} \\ = \frac{1}{2}\delta_n(A_{n,k+1}x_{n,k+1} + A_{n,k}x_{n,k} + p_{n,k+1} + p_{n,k}). \end{aligned} \tag{3.17}$$

Thus, the Haar discretization is similar to the trapezoidal rule applied to the corresponding ordinary differential equation

$$((I + B)x + q)' = Ax + p \tag{3.18}$$

with initial condition

$$(I + B(0))x(0) + q(0) + \lambda = 0. \tag{3.19}$$

Unlike the traditional implementation of the trapezoidal rule, the grid points associated with the above finite-difference interpretation are $\frac{1}{2}\delta_n, \frac{3}{2}\delta_n, \ldots$ and instead of evaluating matrices $A, B, \ldots$ at these points, $A, B, \ldots$ are computed from the projections $\mathbf{P}_n A, \mathbf{P}_n B, \ldots$. Also, the initial step

$$(I + B_{n,0})x_{n,0} + q_{n,0} + \lambda = \frac{1}{2}\delta_n(A_{n,0}x_{n,0} + p_{n,0}) \tag{3.20}$$

corresponds to a backward Euler step with half the step-size.

Let us assume that A and $(I+B)^{-1}$ are bounded on $[0, 1]$. We will show that the hypotheses of Propositions II.1 and II.2 are satisfied. Hence the scheme is convergent; that is,

$$\lim_{n \to -\infty} x^{(n)} = x \tag{3.21}$$

in $\mathbf{L}_2(0, 1)$. If the coefficients are smooth, then it may be shown that the scheme is of second-order accuracy (e.g., because of its similarity to the trapezoidal rule).

Applying the multiresolution strategy of Section II requires evaluation of the expressions in (2.20). In the Haar basis these expressions may be evaluated explicitly. Let us define

$$S^{(n)}_{A,j,k} = \frac{1}{2}(A^{(n)}_{j-1,2k} + A^{(n)}_{j-1,2k+1}), \tag{3.22}$$

and

$$D^{(n)}_{A,j,k} = \frac{2}{\delta_j}(A^{(n)}_{j-1,2k} - A^{(n)}_{j-1,2k+1}), \tag{3.23}$$

and similarly $S^{(n)}_{B,j,k}$ and $D^{(n)}_{B,j,k}$. Further, let us define

$$S^{(n)}_{q,j,k} = \frac{1}{\sqrt{2}}(q^{(n)}_{j-1,2k} + q^{(n)}_{j-1,2k+1}), \tag{3.24}$$

and

$$D^{(n)}_{q,j,k} = \frac{2\sqrt{2}}{\delta_j}(q^{(n)}_{j-1,2k} - q^{(n)}_{j-1,2k+1}), \tag{3.25}$$

and similarly $S^{(n)}_{p,j,k}$ and $D^{(n)}_{p,j,k}$.

Next we evaluate (2.12)–(2.19) in the Haar basis. We note that the operators of the systems (2.11) are

$$\mathbf{A}^{(n)}_j = \mathbf{M}(A^{(n)}_j), \tag{3.26}$$

$$\mathbf{B}^{(n)}_j = \mathbf{M}(B^{(n)}_j), \tag{3.27}$$

where $\mathbf{M}(\cdot)$ is defined in (3.14) and denotes a block-diagonal matrix. The blocks of this matrix, $A^{(n)}_j$ and $B^{(n)}_j$, are generated via a recursion which we derive below. The following table may be verified using the matrix representations, of $\mathbf{K}, \mathbf{A}$ and $\mathbf{B}$. We find

$$\begin{aligned}
\mathcal{T}_{\mathbf{K},j} &= \mathbf{K}_j, & \mathcal{T}^{(n)}_{\mathbf{G},j} &= \operatorname{diag}\{S^{(n)}_{\mathbf{G},j,0}, S^{(n)}_{\mathbf{G},j,1}, \ldots, S^{(n)}_{\mathbf{G},j,N_j-1}\},\\
\mathcal{C}_{\mathbf{K},j} &= \mathbf{I}, & \mathcal{C}^{(n)}_{\mathbf{G},j} &= \operatorname{diag}\{D^{(n)}_{\mathbf{G},j,0}, D^{(n)}_{\mathbf{G},j,1}, \ldots, D^{(n)}_{\mathbf{G},j,N_j-1}\},\\
\mathcal{B}_{\mathbf{K},j} &= -\mathbf{I}, \quad \text{and} & \mathcal{B}^{(n)}_{\mathbf{G},j} &= \operatorname{diag}\{D^{(n)}_{\mathbf{G},j,0}, D^{(n)}_{\mathbf{G},j,1}, \ldots, D^{(n)}_{\mathbf{G},j,N_j-1}\},\\
\mathcal{A}_{\mathbf{K},j} &= 0, & \mathcal{A}^{(n)}_{\mathbf{G},j} &= \operatorname{diag}\{S^{(n)}_{\mathbf{G},j,0}, S^{(n)}_{\mathbf{G},j,1}, \ldots, S^{(n)}_{\mathbf{G},j,N_j-1}\},
\end{aligned} \tag{3.28}$$

where $\mathbf{G}$ is either $\mathbf{A}$ or $\mathbf{B}$. Substitution of (3.28) into (2.20) yields the matrix recurrence relations

$$\begin{aligned}
A &= S_A - \frac{\delta_j^2}{16}D_A F^{-1}(D_B + S_A),\\
B &= S_B - \frac{\delta_j^2}{16}D_A - \frac{\delta_j^2}{16}(D_B - S_A)F^{-1}(D_B + S_A),
\end{aligned} \tag{3.29}$$

and

$$F = I + S_B + \frac{\delta_j^2}{16}D_A, \tag{3.30}$$

where all elements in the above expression have the same indices: j, k and (n). The matrix F is assumed to be invertible. Also, we have

$$\begin{aligned}
p &= S_p - \frac{\delta_j^2}{16}D_A F^{-1}(D_q + S_p),\\
q &= S_q - \frac{\delta_j^2}{16}D_p - \frac{\delta_j^2}{16}(D_B - S_A)F^{-1}(D_q + S_p).
\end{aligned} \tag{3.31}$$

Thus, we have relations to compute $A^{(n)}_{j,k}, B^{(n)}_{j,k}, p^{(n)}_{j,k}$ and $q^{(n)}_{j,k}$ recursively. We now may form equations for projections of $x^{(n)}$ (the solution at the scale of the original discretization) on all multiresolution levels $j \leqslant 0$,

$$\begin{aligned}
(I + B^{(n)}_{j,k})x^{(n)}_{j,k} + q^{(n)}_{j,k} + \lambda &= \frac{1}{2}\delta_j(A^{(n)}_{j,k}x^{(n)}_{j,k} + p^{(n)}_{j,k})\\
&\quad + \delta_j\sum_{k'=0}^{k-1}(A^{(n)}_{j,k'}x^{(n)}_{j,k'} + p^{(n)}_{j,k'}).
\end{aligned} \tag{3.32}$$

These equations have the same structure as those of the original discretization (3.16) at level n. In particular, the operators $\mathbf{A}^{(n)}_j$ and $\mathbf{B}^{(n)}_j$ are block diagonal with blocks of sizes N_x (we recall that N_x is simply the number of equations in the original system of integral equations).

A multiresolution strategy in the Haar representation for obtaining the solution $x^{(n)}$ is a special case of the multiresolution strategy discussed in Section II and is as follows:

1. Construct the sequences $\{A^{(n)}_{j,k}\}, \{B^{(n)}_{j,k}\}, \{p^{(n)}_{j,k}\}, \{q^{(n)}_{j,k}\}$ for $n \leqslant j \leqslant 0$ using (3.29)—(3.31) (see Figs. 1 and 2);

2. Solve for $x^{(n)}_0$ in the subspace $\mathbf{V}_0$ to satisfy the coarse-scale equation (2.25) (see Fig. 3); The elements of $\mathbf{V}_0$ are determined by just one (vector) coefficient. For each n, we obtain

$$(I + B^{(n)}_0)\bar{x}^{(n)} + q^{(n)}_0 + \lambda = \frac{1}{2}(A^{(n)}_0\bar{x}^{(n)} + p^{(n)}_0), \tag{3.33}$$

where $\bar{x}^{(n)}$ is the average of the approximate solution $x^{(n)}$ on the interval of definition $t \in [0, 1]$.

3. Obtain the solution components in the subspaces $\mathbf{W}_n$ by applying (2.26)—that is,

$$v_{j,k}^{(n)} = -C_{j,k}^{(n)} x_j^{(n)} - r_{j,k}^{(n)} \tag{3.34}$$

where

$$C_{j,k}^{(n)} = F_{j,k}^{-1}(S_{A,j,k}^{(n)} + D_{B,j,k}^{(n)}), \tag{3.35}$$

$$r_{j,k}^{(n)} = F_{j,k}^{-1}(D_{q,j,k}^{(n)} + S_{p,j,k}^{(n)}), \tag{3.36}$$

and the reconstruction algorithm

$$x_{j-1,2k}^{(n)} = \frac{1}{\sqrt{2}}\left(x_{j,k}^{(n)} + \frac{\delta_{j,k}}{4} v_{j,k}^{(n)}\right),$$

$$x_{j-1,2k+1}^{(n)} = \frac{1}{\sqrt{2}}\left(x_{j,k}^{(n)} - \frac{\delta_j}{4} v_{j,k}^{(n)}\right). \tag{3.37}$$

(See Fig. 4.)

The first step sweeps through the resolution levels from finest to coarsest, and involves calculations on the coefficients and forcing terms of the equations. The intermediate calculations, that is $A_{j,k}^{(n)}, B_{j,k}^{(n)}, p_{j,k}^{(n)}, q_{j,k}^{(n)}, j < 0$ need not be saved except as they contribute to the terms in (3.34). The second step is the solution of a single linear system of size N_x, the number of integral equations in the system (2.2), to determine the solution component at the coarsest level. The third step sweeps through the resolution levels from coarsest to finest, determining solution components explicitly.

The strategy as described above carries the analysis up to $\mathbf{V}_0$, the coarsest level possible. We may also stop the analysis at any specified level, say $\mathbf{V}_j$. In this case to obtain the solution we have Eq. (3.32). We note that the system of equations (3.32) is a set of N_j implicit systems of equations.

Linear Functionals of the Solution. In the Haar case the computation of linear functionals of the solution (see Section II) may be made explicit. Let us consider the calculation of the endpoint value $x(1)$. From (3.4), (3.11) and (3.37) we have

$$x^{(n)}(1) = \frac{1}{\sqrt{\delta_n}} x_{n,N_n-1}^{(n)} = \frac{1}{\sqrt{\delta_{n+1}}}\left(x_{n+1,N_{n+1}-1}^{(n)} - \frac{\delta_{n+1}}{4} v_{n+1,N_{n+1}-1}^{(n)}\right). \tag{3.38}$$

Using (3.34) we obtain

$$x^{(n)}(1) = \frac{1}{\sqrt{\delta_{n+1}}}(\alpha_{n+1}^{(n)} x_{n+1,N_{n+1}-1}^{(n)} + \beta_{n+1}^{(n)}), \tag{3.39}$$

where

$$\alpha_{n+1}^{(n)} = 1 + \frac{\delta_{n+1}}{4} C_{n+1,N_{n+1}-1}^{(n)}, \tag{3.40}$$

$$\beta_{n+1}^{(n)} = \frac{\delta_{n+1}}{4} r_{n+1,N_{n+1}-1}^{(n)}. \tag{3.41}$$

Repeating this process for each reduction step, we arrive at

$$x^{(n)}(1) = \frac{1}{\sqrt{\delta_j}}(\alpha_j^{(n)} x_{j,N_j-1}^{(n)} + \beta_j^{(n)}), \tag{3.42}$$

where

$$\alpha_n^{(n)} = 1, \quad \beta_n^{(n)} = 0, \tag{3.43}$$

and

$$\alpha_j^{(n)} = \alpha_{j-1}^{(n)} + \frac{\delta_j}{4} C_{j,N_{j-1}}^{(n)}, \tag{3.44}$$

$$\beta_j^{(n)} = \beta_{j-1}^{(n)} + \frac{\delta_j}{4} r_{j,N_{j-1}}^{(n)}. \tag{3.45}$$

Other linear functionals may be treated in a similar manner.

Remark. By subtracting successive steps in time of (3.32) we get the compact scheme (which is similar to a finite-difference scheme)

$$\left(I + B_{j,k+1}^{(n)} - \frac{1}{2}\delta_j A_{j,k+1}^{(n)}\right) x_{j,k+1}^{(n)} = \left(I + B_{j,k}^{(n)} + \frac{1}{2}\delta_j A_{j,k}^{(n)}\right) x_{j,k}^{(n)} + q_{j,k}^{(n)} - q_{j,k+1}^{(n)} + \frac{1}{2}\delta_j(p_{j,k+1}^{(n)} + p_{j,k+1}^{(n)}). \tag{3.46}$$

This scheme has a step size δ_j and global accuracy of $O(\delta_n^2)$ where δ_n is the step size of the initial discretization. Since we can make δ_n arbitrarily small while keeping δ_j fixed, we can view the resulting scheme as having arbitrary (finite) order.

We will now analyze the recurrence relations (3.29)–(3.30). The following lemma will be used to verify that Propositions 1 and 2 apply to the Haar multiresolution strategy. For simplicity we consider the case of (2.2) where $B = 0$. A similar result may be obtained if B is a Lipschitz function.

LEMMA 1. *Consider the recurrence relations* (3.29)–(3.30) *where* $B_{n,k}^{(n)} = 0$ *and assume*

$$\|\mathbf{A}_{n,k}^{(n)}\| \leq \alpha \tag{3.47}$$

for all k, *where* $\|\cdot\|$ *denotes a matrix norm. Then*

$$\|\mathbf{A}_{j,k}^{(n)}\| \leq \frac{\alpha}{1 - 2^j(\beta + \alpha/2)}, \tag{3.48}$$

$$\|B_{j,k}^{(n)}\| \leq 2^j \beta, \tag{3.49}$$

where $0 < \beta < 1$ satisfies

$$\alpha = \frac{2\beta(2-\beta)(1-\beta)}{2+3\beta-\beta^2}. \tag{3.50}$$

The maximum value of α is ≈ 0.2564 and is attained for $\beta \approx 0.3473$.

Proof. Let

$$\theta_j = 2^j\left(\beta + \frac{\alpha}{2}\right), \tag{3.51}$$

$$\alpha_j = \frac{\alpha}{1-\theta_j}, \tag{3.52}$$

$$\beta_j = 2^j\beta, \tag{3.53}$$

$$\gamma_j = 2^j\left(\beta + \frac{\alpha}{2(1-\theta_j)}\right) = \beta_j + 2^{j-1}\alpha_j. \tag{3.54}$$

We proceed by induction. Note that the inequalities (3.48) and (3.49) are automatically satisfied for $j = n$ by hypothesis. Now suppose that (3.48) and (3.49) are satisfied at level $j - 1 < 0$. That is,

$$\|\mathbf{A}^{(n)}_{j-1,k}\| \leq \alpha_{j-1}, \tag{3.55}$$

$$\|\mathbf{B}^{(n)}_{j-1,k}\| \leq \beta_{j-1}. \tag{3.56}$$

We observe from (3.22) that

$$\|S^{(n)}_{A,j,k}\| \leq \alpha_{j-1}, \tag{3.57}$$

$$\|2^{j-1}D^{(n)}_{A,j,k}\| \leq \alpha_{j-1}, \tag{3.58}$$

and, similarly,

$$\|S^{(n)}_{B,j,k}\| \leq \beta_{j-1}, \tag{3.59}$$

$$\|2^{j-1}D^{(n)}_{B,j,k}\| \leq \beta_{j-1}. \tag{3.60}$$

Using the inequality

$$\|(I+M)^{-1}\| \leq \frac{1}{1-\|M\|} \tag{3.61}$$

for $\|M\| < 1$, we find from (3.30)

$$\|F_{j,k}\| \leq \frac{1}{1-\beta_{j-1}-2^{j-2}\alpha_{j-1}} \leq \frac{1}{1-\gamma_{j-1}} \tag{3.62}$$

provided

$$\gamma_{j-1} < 1. \tag{3.63}$$

Therefore, we obtain from (3.29)

$$\|A^{(n)}_{j,k}\| \leq \alpha_{j-1} + \frac{\alpha_{j-1}\gamma_{j-1}}{1-\gamma_{j-1}} = \frac{\alpha_{j-1}}{1-\gamma_{j-1}}, \tag{3.64}$$

$$\|B^{(n)}_{j,k}\| \leq \gamma_{j-1} + \frac{\gamma^2_{j-1}}{1-\gamma_{j-1}} = \frac{\gamma_{j-1}}{1-\gamma_{j-1}}. \tag{3.65}$$

Now using the definition of α_{j-1}, we find

$$\frac{\alpha_{j-1}}{1-\gamma_{j-1}} = \frac{\alpha}{1-(\theta_{j-1}+\gamma_{j-1}(1-\theta_{j-1}))}. \tag{3.66}$$

From (3.51) and (3.54) we also have

$$\theta_{j-1} + \gamma_{j-1}(1-\theta_{j-1}) = 2^{j-1}\left(\beta + \frac{\alpha}{2}\right) + 2^{j-1}\left(\beta(1-\theta_{j-1}) + \frac{\alpha}{2}\right) \leq 2^j\left(\beta + \frac{\alpha}{2}\right) = \theta_j.$$

Combining (3.64), (3.66) and (3.67), we obtain

$$\|A^{(n)}_{j,k}\| \leq \alpha_j \tag{3.67}$$

as required.

We now turn to (3.65). To show that

$$\|B^{(n)}_{j,k}\| \leq \beta_j \tag{3.68}$$

we will demonstrate that

$$\frac{\gamma_{j-1}}{1-\gamma_{j-1}} \leq \frac{\gamma_{-1}}{1-\gamma_{-1}} \tag{3.69}$$

and

$$\frac{\gamma_{-1}}{1-\gamma_{-1}} = \beta. \tag{3.70}$$

From (3.51) we obtain

$$\theta_{j-1} \leq \theta_{-1} = \frac{\beta}{2} + \frac{\alpha}{4} < 1, \tag{3.71}$$

the last estimate being easily verified for $0 < \beta < 1$. Hence we have

$$\frac{1}{1-\theta_{j-1}} \leq \frac{1}{1-\theta_{-1}}, \tag{3.72}$$

and

$$\gamma_{j-1} = 2^{j-1}\left(\beta + \frac{\alpha}{1-\theta_{j-1}}\right) \leq 2^{j-1}\left(\beta + \frac{\alpha}{1-\theta_{-1}}\right) = 2^{j-1}\gamma_{-1}, \tag{3.73}$$

and, therefore, (3.69) follows. Using (3.54) and (3.50) one may verify (3.70). It follows from

$$\gamma_{j-1} \leqslant \gamma_{-1} = \frac{\beta}{1+\beta} < 1 \tag{3.74}$$

that the condition (3.63) is met.

To demonstrate the convergence of our scheme, we will show now that Propositions II.1 and II.2 apply to the Haar multiresolution strategy for the integral equation (2.2) under mild restrictions on the coefficients. Let us take $B = 0$ for simplicity. All multiplication and integral operators in (2.2) are bounded. By the standard theory of linear differential equations, the initial-value problem has a unique solution. Thus hypotheses of Proposition II.1 are satisfied. Then provided

$$|A(t)| \leqslant \alpha, \tag{3.75}$$

where α is given in (3.50), Lemma 1 implies that the sequences $A_j^{(n)}, B_j^{(n)}, p_j^{(n)}, q_j^{(n)}$ are well-defined and bounded. For a fixed j, there is a convergent subsequence for which the limits

$$\lim_{n\to-\infty} A_j^{(n)} = A_j^{(-\infty)}, \quad \lim_{n\to-\infty} B_j^{(n)} = B_j^{(-\infty)},$$
$$\lim_{n\to-\infty} q_j^{(n)} = q_j^{(-\infty)}, \quad \lim_{n\to-\infty} p_j^{(n)} = p_j^{(-\infty)} \tag{3.76}$$

exist (double-indices have been dropped for convenience). Consequently, the hypotheses of Proposition II.2 are satisfied and, thus the conclusions of Proposition II.2 are applicable to subsequences in (3.76). For example, we have for $j = 0$,

$$(I + B_0^{(-\infty)})\bar{x} + q_0^{(-\infty)} = \frac{1}{2}(A_0^{(-\infty)}\bar{x} + p_0^{(-\infty)}), \tag{3.77}$$

where $\bar{x}$ is the average of the exact solution $x(t)$ on the interval of definition $t \in [0, 1]$. An illustration of what may go wrong if hypothesis (3.75) fails will be given in Section V.

To summarize: we have

1. developed a multiresolution strategy for solving linear systems, as outlined in (3.29)–(3.37);

2. constructed a compact scheme (3.46) of arbitrary order.

4. HOMOGENIZATION

The multiresolution strategy described in the previous section was motivated as a technique to obtain solutions or a Green's function for the given equation. Another perspective which appears to follow naturally from our multiresolution analysis is that of homogenization.

The homogenization problem associated with (2.2) is the task of finding an integral equation with "slowly-varying" coefficients whose solutions have the same coarse-scale behavior as those of the original equation. Within the multiresolution strategy developed for the Haar case, this means finding an integral equation of the form (2.2) with piecewise-constant coefficients which are elements of a coarse subspace of the MRA. We already have derived in (3.32) a discrete system of equations which describe the behavior on coarse subspaces $\mathbf{V}_j$. Convergence shown in the previous section implies that

$$(I + B_{j,k}^{(-\infty)})x_{j,k} + q_{j,k}^{(-\infty)} + \lambda = \frac{1}{2}\delta_j(A_{j,k}^{(-\infty)}x_{j,k} + p_{j,k}^{(-\infty)}) + \delta_j \sum_{k'=0}^{k-1}(A_{j,k'}^{(-\infty)}x_{j,k'} + p_{j,k'}^{(-\infty)}), \tag{4.1}$$

where $x_{j,k}$ are the coefficients of the projection of the exact solution on the subspace $\mathbf{V}_j$.

Let us now consider a problem of finding an integral equation of the form (2.2) with coefficients in $\mathbf{V}_j$ whose discretization yields (4.1). Let us consider $j = 0$ in which case (4.1) becomes (3.77), namely,

$$(I + B_0^{(-\infty)})\bar{x} + q_0^{(-\infty)} + \lambda = \frac{1}{2}(A_0^{(-\infty)}\bar{x} + p_0^{(-\infty)}), \tag{4.2}$$

where the constant vector $\bar{x}$ is the average of the exact solution $x(t)$ on the interval of definition $t \in [0, 1]$.

We wish to determine "homogenized" coefficients A^h, B^h and forcing terms q^h, p^h in the integral equation

$$(I + B^h)x(t) + q^h + \lambda = \int_0^t (A^h x(s) + p^h(s))\, ds, \tag{4.3}$$

such that applying exactly the same procedure to (4.3) as was applied to (2.2) in order to obtain (4.2) yields the same result for all λ.

The recurrence relations (3.29), (3.31) applied to (4.3) simplify to

$$\begin{aligned} A_j^h &= A_{j-1}^h \\ B_j^h &= B_{j-1}^h + \frac{\delta_j^2}{16}A_{j-1}^h(I + B_{j-1}^h)^{-1}A_{j-1}^h \\ q_j^h &= q_{j-1}^h - \frac{\delta_j^2}{16}A_{j-1}^h(I + B_{j-1}^h)^{-1}p_{j-1}^h \\ p_j^h &= p_{j-1}^h. \end{aligned} \tag{4.4}$$

Clearly, the matrix coefficient A_j^h and the forcing term p_j^h remain unchanged so we will delete the subscript from now on. Thus the homogenized coefficient A^h is given by $A^h = A_0^{(-\infty)}$ and the homogenized forcing is $p^h = p_0^{(-\infty)}$. It remains to determine the homogenized coefficient B^h and the inhomogeneity q^h which are not in general the same as $B_0^{(-\infty)}$ and $q_0^{(-\infty)}$. This can be carried out analytically using the exact solution of (4.3).

First we take the case where $p^h = p_0^{(-\infty)} = 0$. Then it follows from (4.4) that $q^h = q_0^{(-\infty)}$. The solution of (4.3) is

$$x(t) = -\exp(\tilde{A}t)\tilde{q} \tag{4.5}$$

where

$$\tilde{A} = (I + B^h)^{-1}A^h, \quad \tilde{q} = (I + B^h)^{-1}(q^h + \lambda). \tag{4.6}$$

The average of the solution of (4.3) is thus

$$\bar{x} = \left(-\int_0^1 \exp(\tilde{A}t)\,dt\right)\tilde{q} = (I - \exp(\tilde{A}))\tilde{A}^{-1}\tilde{q}. \tag{4.7}$$

(If $\tilde{A}$ is singular, then one needs to use an expansion to avoid $\tilde{A}^{-1}$.) But we can also solve (4.2) for $\bar{x}$. We find

$$\bar{x} = -\left(I + B_0^{(-\infty)} - \frac{1}{2}A_0^{(-\infty)}\right)^{-1}\left(q_0^{(-\infty)} + \lambda\right). \tag{4.8}$$

The equivalence of (4.7) and (4.8) must hold for all λ. Thus we have the relation

$$\left(I + B_0^{(-\infty)} - \frac{1}{2}A^h\right)^{-1} = (\exp(\tilde{A}) - I)\tilde{A}^{-1}(I + B^h)^{-1}, \tag{4.9}$$

where we have replaced $A_0^{(-\infty)}$ by A^h. Solving for B^h in terms of $B_0^{(-\infty)}$ and A^h

$$B^h = A^h\tilde{A}^{-1} - I \tag{4.10}$$

where

$$\tilde{A} = \log\left(I + \left(I + B_0^{(-\infty)} - \frac{1}{2}A^h\right)^{-1}A^h\right). \tag{4.11}$$

It is also useful to have an expression for $B_0^{(-\infty)}$ in terms of B^h and A^h yields

$$B_0^{(-\infty)} = A^h(\exp((I + B^h)^{-1}A^h) - I)^{-1} + \frac{1}{2}A^h - I. \tag{4.12}$$

Secondly, we consider the case when $q_0^{(-\infty)} = 0$ and $p_0^{(-\infty)} \neq 0$ to determine the contribution to q^h from p^h. The solution in this case is

$$x(t) = (\exp(\tilde{A}t) - I)(A^h)^{-1}p^h - \exp(\tilde{A}t)\tilde{q}, \tag{4.13}$$

and the average is

$$\bar{x} = ((\exp(\tilde{A}) - I)\tilde{A}^{-1} - I)(A^h)^{-1}p^h - (\exp(\tilde{A}) - I)\tilde{A}^{-1}\tilde{q}. \tag{4.14}$$

Also solving (4.2) for $\bar{x}$ we find

$$\bar{x} = \frac{1}{2}\left(I + B_0^{(-\infty)} - \frac{1}{2}A^h\right)^{-1}p^h. \tag{4.15}$$

Comparing (4.14) and (4.15) and solving for q^h gives

$$q^h = \left(A^h\tilde{A}^{-1}(A^h)^{-1} - \frac{1}{2}I - A^h(\exp(\tilde{A}) - I)^{-1}(A^h)^{-1}\right)p^h. \tag{4.16}$$

Finally, combining the contributions from $q_0^{(-\infty)}$ and p^h we obtain the formula for the general case, $q_0^{(-\infty)} \neq 0$ and $p_0^{(-\infty)} \neq 0$,

$$q^h = q_0^{(-\infty)} + \left(A^h\tilde{A}^{-1}(A^h)^{-1} - \frac{1}{2}I - A^h(\exp(\tilde{A}) - I)^{-1}(A^h)^{-1}\right)p^h. \tag{4.17}$$

Remark 1. There is no loss of generality by considering the case $j = 0$. To see this, let us first note that the above analysis may be carried out for intervals other than of unit length. If the interval has a length $L \neq 1$, then Eqs. (4.10), (4.11) should be modified by replacing A^h with LA^h. Secondly, we may perform the homogenization procedure up to any level j (not necessarily $j = 0$), say $\mathbf{V}_j$. We note that since functions of the Haar basis on a given subspace $\mathbf{V}_j$ have non-overlapping supports, we simply perform our analysis on subintervals of length δ_j.

Remark 2. The homogenization procedure above preserves the average of the solution on specified intervals. Alternatively, we may preserve a linear functional of the solution, for example higher moments, e.g.,

$$\int_0^1 tx(t)\,dt, \tag{4.18}$$

or the endpoint value $x(1)$ by generalizing the procedure above. The resulting homogenized equation depends on the choice of the linear functional.

In summary, in addition to the multiresolution solution strategy outlined in Section III, we now have a method for generating homogenized equations of the form (2.2) that preserve specified linear functionals of the solution for arbitrary initial conditions. We consider examples in Section V.

5. EXAMPLES

Example 1. In the first examples we will evaluate the accuracy of our numerical method using a simple constant-coefficient equation

$$x(t) + 1 = i\alpha \int_0^t x(s)\,ds \quad \text{for } t \geq 0 \tag{5.1}$$

whose solution is known analytically. For our calculations we choose $\alpha = 1.5$. Discretization is applied on the space $\mathbf{V}_n, n < 0$ in which the mesh has a step-size $\delta_n = 2^n$. We will investigate the error in the resulting scheme as a function of t for a fixed value of n, and further as a function of n.

The computation is performed as follows: the initialization step (2.24) becomes

$$A_{n,k}^{(n)} = i\alpha, \quad B_{n,k}^{(n)} = 0. \tag{5.2}$$

We notice that there is no dependence on k and, thus, averages and differences defined in (3.22) and (3.23) simplify to

$$\begin{aligned} S_{A,j}^{(n)} &= A_{j-1}, & S_{B,j}^{(n)} &= B_{j-1}, \\ D_{A,j}^{(n)} &= 0, & D_{B,j}^{(n)} &= 0. \end{aligned} \tag{5.3}$$

Therefore, the recurrence relations (3.29)–(3.30) yield

$$A_j = A_{j-1} = i\alpha, \quad B_j = B_{j-1} + \frac{\delta_j^2 (A_{j-1})^2}{16(1 + B_{j-1})}. \tag{5.4}$$

These recurrence relations are applied $-n$ times ($n < 0$), yielding the coefficients of the reduced scheme (3.46)

$$\left(I + B_0^{(n)} - \frac{1}{2} i\alpha\right) x_{0,k+1}^{(n)} = \left(I + B_0^{(n)} + \frac{1}{2} i\alpha\right) x_{0,k}^{(n)} \tag{5.5}$$

with step size $\delta_0 = 1$. This step size is approximately one-quarter of a period of the oscillator (5.1).

Then we solve the implicit reduced scheme to obtain the projection of the solution on $\mathbf{V}_0$ with corresponding step-size $\delta_0 = 1$. The error is obtained by comparing the results of the computation with the exact solution of (5.1) on the interval $0 \leq t \leq 64$. The numerical results are shown in Figs. 5 and 6. The real and imaginary parts of the error are plotted in Figs. 5 and 6 for $n = -11, -12$, respectively. Note that the amplitude of the dispersive (phase) error grows linearly. The magnitude of the error for fixed t goes down by a factor of 4 as n decreases by one showing the quadratic dependence of the error on the fine-scale mesh-size, $\delta_n = 2^n$. We emphasize however that in all cases the computation is done via the reduced scheme (3.46) with step size $\delta_0 = 1$.

The reduced scheme (3.46) has a form similar to a finite difference scheme. Finite difference schemes, even those of high order, applied to problems whose solutions have oscillating behavior are known to exhibit dispersive errors (that is, errors in the phase of the oscillation) on large time intervals. For this reason, spectral methods are often preferred for such problems. We now demonstrate that it is possible to achieve a discretization error in the reduced scheme (3.46) limited by machine precision by letting $-n$ be sufficiently large. By letting $n = -1, \ldots, -26$ we observe numerically the convergence of the coefficients of the reduced scheme for the equation (5.1). From (4.12) we find the limiting values as n tends to $-\infty$ of the coefficients of the reduced scheme

$$B_0^{(n)} \to B_0^{(-\infty)} = \frac{i\alpha}{(\exp(i\alpha) - 1)} + \frac{i\alpha}{2} - 1. \tag{5.6}$$

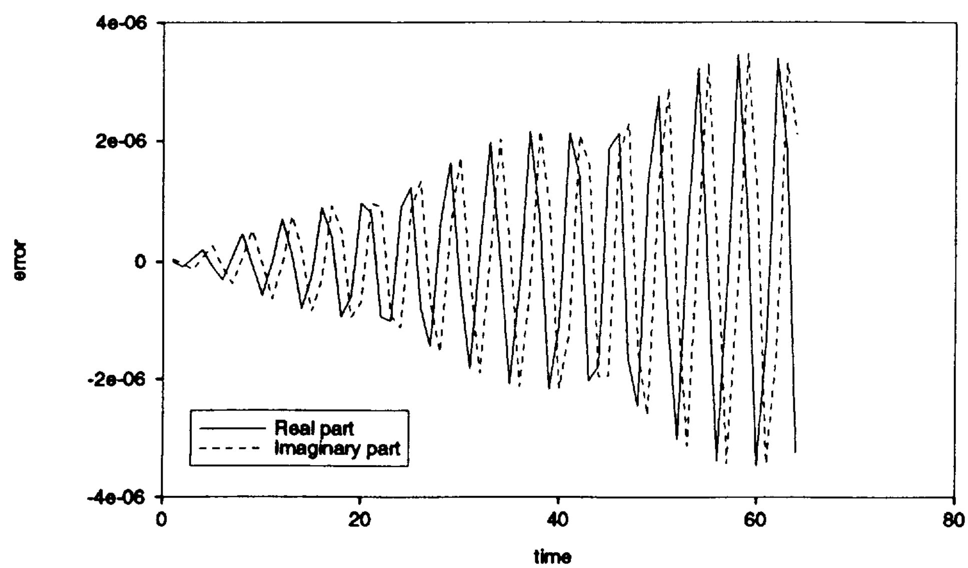

FIG. 5. Behavior of error for one-way wave equation.

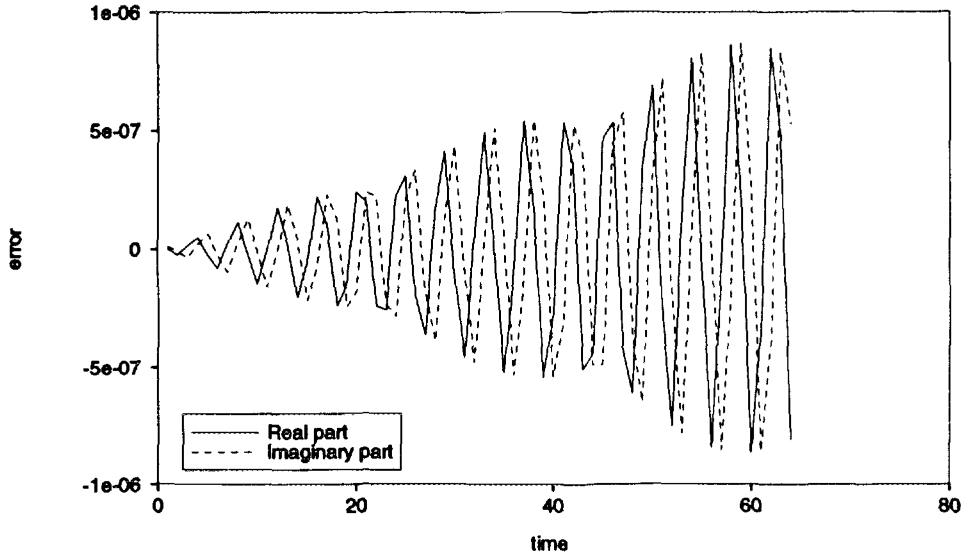

FIG. 6. Same as in Fig. 5 but with one extra level.

Thus, we obtain

$$\lim_{n\to-\infty} \frac{I + B_0^{(n)} + \frac{1}{2}i\alpha}{I + B_0^{(n)} - \frac{1}{2}i\alpha} = e^{i\alpha}, \tag{5.7}$$

which explains the improved accuracy of the scheme. We note that in practice n is large but finite.

For $n = -26$ the error of the coefficients when compared with the exact value is on the order of 10^{-17}. We then used the reduced scheme with $n = -26$ to compute the solution (5.1) over very large time intervals. The error in the numerical solution on the interval $0 \leqslant t \leqslant 20$, approximately 250,000 periods of the oscillator, when compared with the exact solution of (5.1) was less than 10^{-8}. Thus the dispersive error of the reduced scheme (3.46) can be essentially eliminated by letting $-n$ be sufficiently large while keeping the step size fixed.

EXAMPLE 2. In our second example, we study a scalar variable-coefficient equation with complex coefficients

$$(1 + b(t))x(t) + 1 = i\int_0^t \alpha(s)x(s)\,ds, \tag{5.8}$$

where α and b are real-valued. The solution has the form of a right-travelling wave with variable frequency and amplitude. The function α is chosen to model a two-phase material. On each dyadic interval at the finest level of discretization we let $\alpha(s)$ assume one of two values, α_1 or α_2, chosen at random with probabilities p and $1 - p$, respectively. Thus α is an element of $\mathbf{V}_n$ by construction. We choose $b(t) = -\beta t(1 - t)$ to be a continuous function in order to have a continuous solution.

Discretization is applied on the space $\mathbf{V}_n, n < 0$ in which the mesh has step-size $\delta_n = 2^n$. The computation is performed as follows: the recurrence relations (3.29)—(3.30) are applied $-n + j$ times, yielding the coefficients of the reduced scheme (3.46) with step size $\delta_j = 2^j$. Then the homogenization procedure (4.10) is applied to give an equation of the form (5.8) with coefficients A^h, B^h in $\mathbf{V}_j$. A solution of an equation of the form (5.8) with homogenized coefficients A^h, B^h has an identical projection into the subspace $\mathbf{V}_j$ as the solution of the original problem (5.8). For this calculation we set $\alpha_1 = 0.1, \alpha_2 = 0.333\ldots, p = 0.25, \beta = 0.4$ and $n = -6$. In Fig. 7 we have the original coefficient $a = i\alpha(t)$, which is an element of $\mathbf{V}_n$, and the projection of b into the space $\mathbf{V}_n$. In Figs. 8–11 the coefficients are shown for $j = -4, -2$ which corresponds to 2 and 4 levels of homogenization. After 6 levels of homogenization, the coefficients are constants and are given by

$$\begin{aligned} a^h &= 0.0000271967760 + i0.05215346709809,\\ b^h &= -0.06759470188503 - i0.0004862688849. \end{aligned} \tag{5.9}$$

EXAMPLE 3. In our third example we consider the propagation of planar waves in a stratified medium, a classical problem (see e.g. [2]) that was recently studied in [7]. The governing system of equations is

$$(\lambda + 2\mu)w_z + p_t = 0, \quad p_z + \rho w_t = 0. \tag{5.10}$$

Assuming solutions of the form

$$w = e^{-i\Omega t}y_1(z), \quad p = e^{-i\Omega t}y_2(z), \tag{5.11}$$

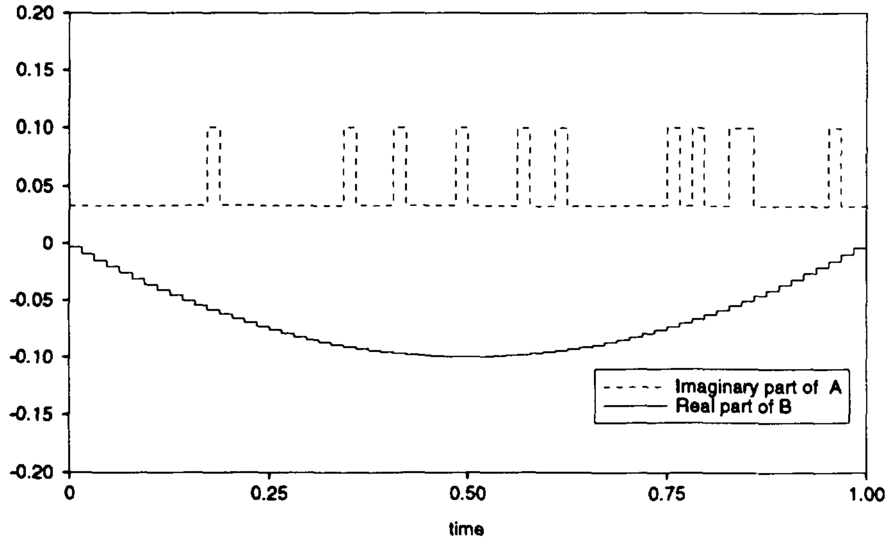

FIG. 7. Original coefficients projected on the level −6.

we obtain the system

$$y_1' = i\Omega(\lambda + 2\mu)^{-1}y_2, \quad y_2' = i\Omega\rho y_1. \tag{5.12}$$

Here Ω represents the frequency of the propagating wave. In integral form the equations (5.12) become

$$\mathbf{y}(z) - \mathbf{y}_0 = i\Omega \int_0^z A(s)\mathbf{y}(s)\,ds, \tag{5.13}$$

where

$$A = \begin{pmatrix} 0 & (\lambda + 2\mu)^{-1} \\ \rho & 0 \end{pmatrix}. \tag{5.14}$$

Periodic Medium. Now consider the homogenization problem associated with (5.13) where the coefficient matrix $A(z)$ is periodic. Without loss of generality we assume the period is 1. We wish to find a constant coefficient equation

$$\mathbf{y}(z) - \tilde{\mathbf{y}}_0 = iK(\Omega) \int_0^z \mathbf{y}(s)\,ds \tag{5.15}$$

whose solution has the same projection on the subspace $\mathbf{V}_0$ as a solution of (5.13). Thus the solution of (5.13) and (5.15) have the same average on [0, 1]. This problem is slightly different than the homogenization problem treated in the previous section since we allow for a change in the initial condition $\tilde{\mathbf{y}}_0$ but fix the coefficient on the $\mathbf{y}(z)$ term as

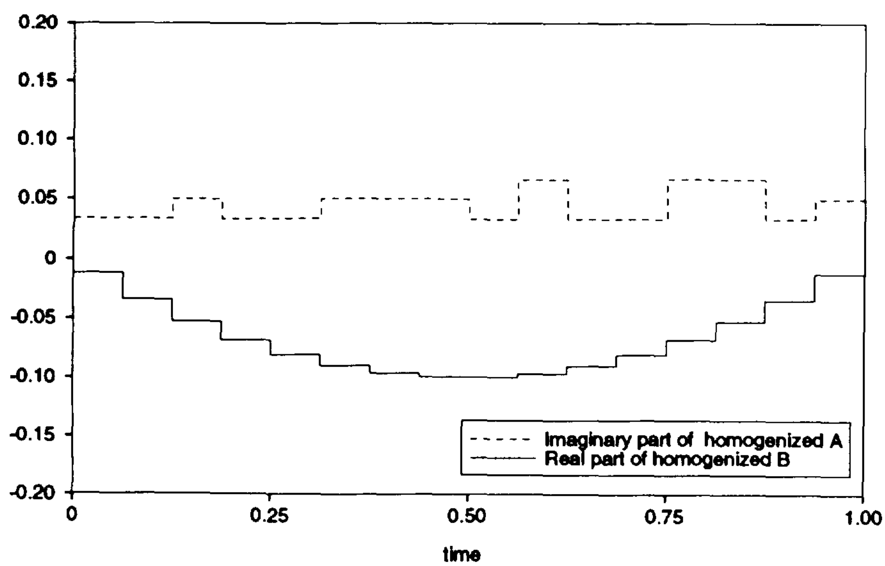

FIG. 8. Coefficients of the reduced equation at level −4. This corresponds to 2 levels of reduction.

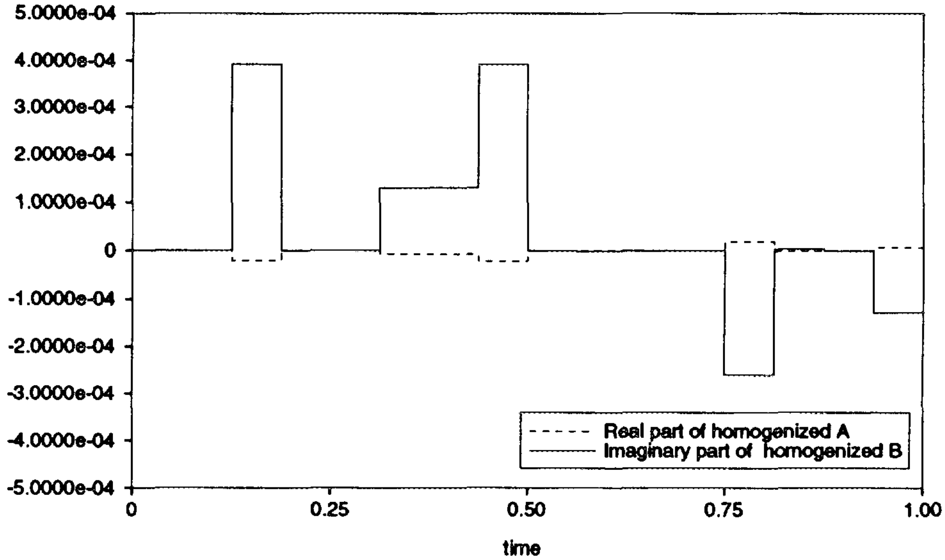

FIG. 9. Coefficients of the reduced equation at level −4. This corresponds to 2 levels of reduction.

the identity. The methods of the Section IV yield a reduced scheme

$$(I + B_0^{(-\infty)})\bar{\mathbf{y}} - \mathbf{y}_0 = \frac{1}{2}A_0^{(-\infty)}\bar{\mathbf{y}}, \tag{5.16}$$

where $\bar{\mathbf{y}}$ is the average of $\mathbf{y}(z)$ on [0, 1]. The values $A_0^{(n)}, B_0^{(n)}$ are obtained from the recursion relations (3.29)–(3.30), and passage to the limit as n tends to $-\infty$ yields the limiting values $A_0^{(-\infty)}, B_0^{(-\infty)}$. The methods of Section IV yield a homogenized equation of the form

$$(I + B^h)\mathbf{y}(z) - \mathbf{y}_0 = A^h \int_0^z \mathbf{y}(s)\,ds, \tag{5.17}$$

whose solution has the same projection on the subspace $\mathbf{V}_0$ as the solution of (5.13). The homogenized coefficients $A^h = A_0^{(-\infty)}, B^h$ are obtained from the coefficients of (5.16) via (4.10). Thus we may obtain the 2×2 matrix coefficient $K(\Omega)$ of equation (5.15) for a fixed value of Ω by obtaining $A^h(\Omega), B^h(\Omega)$, and then applying the formula

$$K = -i(I + B^h)^{-1}A^h, \tag{5.18}$$

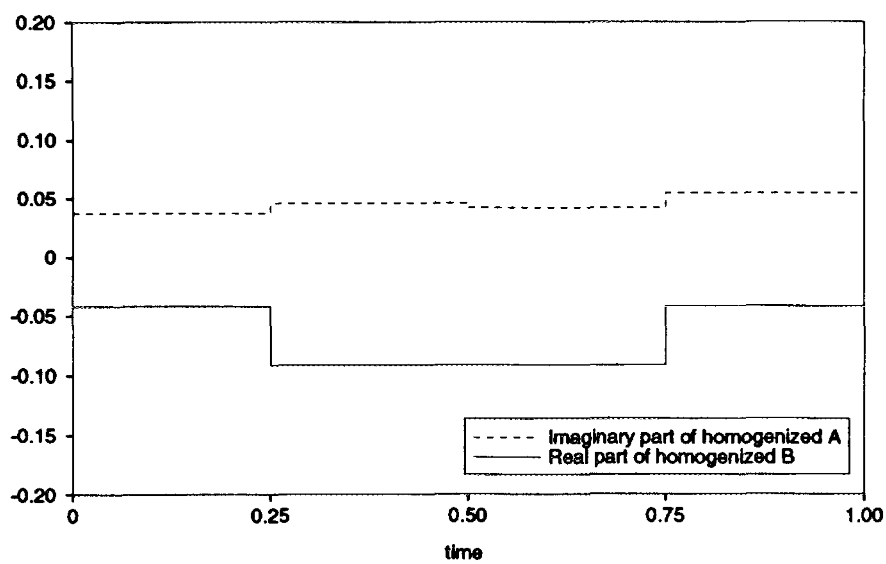

FIG. 10. Coefficients of the reduced equation at level −2. This corresponds to 4 levels of reduction.

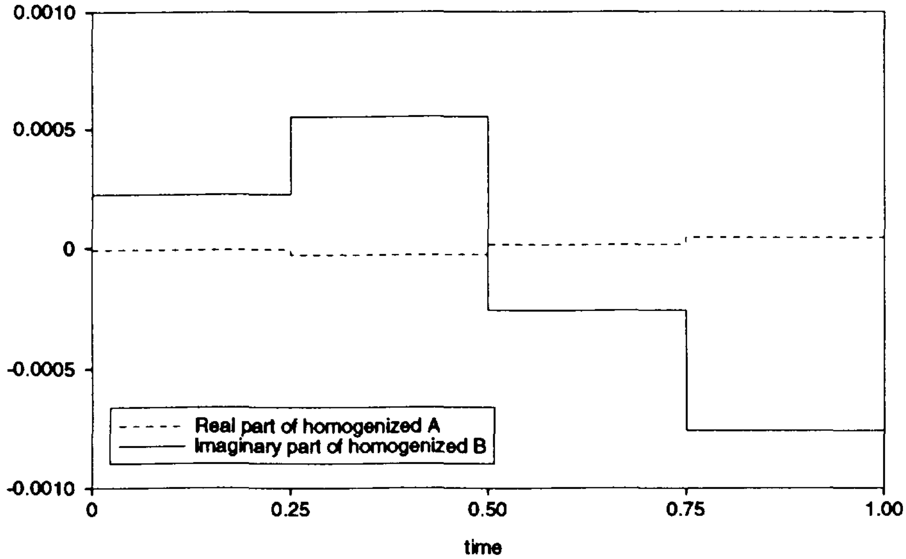

FIG. 11. Coefficients of the reduced equation at level -2. This corresponds to 4 levels of reduction.

provided the inverse exists. Similarly, we find

$$\tilde{\mathbf{y}}_0 = (I + B^h)^{-1}\mathbf{y}_0. \tag{5.19}$$

The matrix K given by formula (5.18) is not a unique solution to the homogenization problem. Figure 12 shows how two sinusoidal oscillations whose circular frequencies differ by $2\pi/L$ give identical averages on intervals of length L (see Fig. 13), an effect sometimes referred to as aliasing. Thus adding to K any matrix whose eigenvalues are multiples of $2\pi i$ yields another solution to the homogenization problem ($L = 1$ in this example). This multiplicity can also be linked to the multiplicity of the logarithm in (4.10).

To avoid aliasing, we choose wave numbers in the interval $[0, \pi/L]$, a convention introduced in [2]. It is sufficient in our problems to choose the frequency of a sinusoidal solution in a range of length π/L rather than $2\pi/L$ because of the existence of right- and left-travelling waves having the same speed of propagation.

In [7], two asymptotic theories are compared. One of the asymptotic theories is based on the limit of small Ω, the other on the limit of small amplitude variation of the coefficients. Our numerical calculations require neither assumption, although the theory, discussed in the previous section, requires Ω to be sufficiently small in order to guarantee that the homogenization coefficients A^h, B^h are bounded. In the

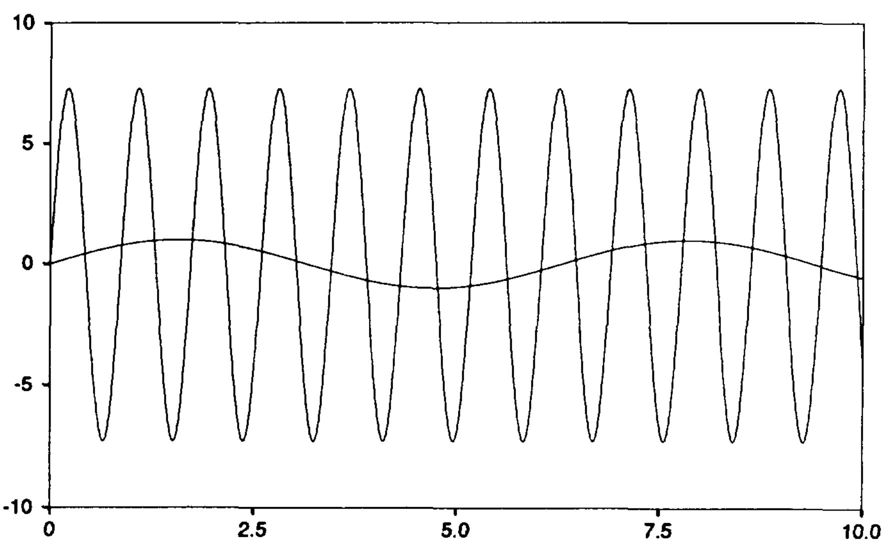

FIG. 12. Two sinusoidal functions which have the same projection on the coarsest scale $\mathbf{V}_0$.

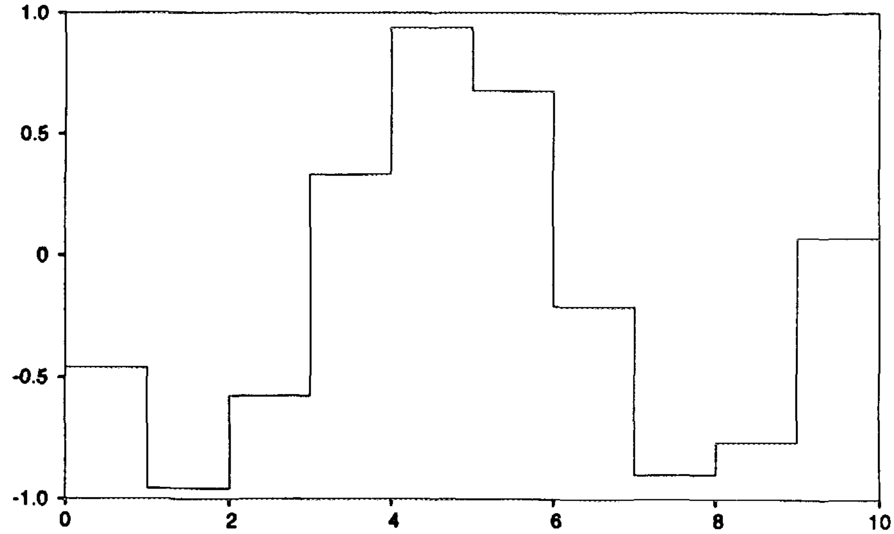

FIG. 13. Projection on the coarsest scale V_0 of the two functions in Fig. 12.

appropriate parameter regimes, we may compare our results to the asymptotic results for validation purposes.

According to the asymptotic theory, for sufficiently small values of Ω the eigenvalues of K are real and are of equal magnitude with opposite sign, $\pm\kappa(\Omega)$ and represent the wave number of a travelling-wave solution. The dependence of the wave number, $\kappa(\Omega)$, of a travelling wave solution on frequency is known as the dispersion relation. A nonlinear dependence of κ on Ω produces a wave velocity that depends on frequency; thus, a signal made up of multiple frequencies gradually spreads as it passes through a material with a nonlinear dispersion relation. For two-phase materials, this effect is attributed to multiple reflections [3]. The effect may also be observed in materials with continuously varying properties.

It has also been observed (see [2], and references therein), that there may be frequency intervals in which the wave number becomes complex. That is, the wave amplitude grows or decays exponentially with respect to the spatial variable. Such frequency intervals are often called stopping bands. Physically, an incoming wave is totally reflected from the material if the frequency is in the stopping band. Intervals of frequency in which the wave number is real are called passing bands.

Two-Phase Material. For this calculation, we choose a two-phase material with phases having material parameters $\rho_1 = 2, \rho_2 = 10^{-4}$, and $(\lambda + 2\mu)_1^{-1} = 10^{-4}, (\lambda + 2\mu)_2^{-1} = 2$. The variation in material parameters is exaggerated to emphasize the nonlinearity of the dispersion relation. The material is assumed to be periodic with period 1 and to have alternating layers of width $\frac{1}{2}$. We now calculate the dispersion relation for this medium. In Fig. 14, we show the dependence of the coefficients $A_0^{(-\infty)}, B_0^{(-\infty)}$ of the reduced scheme (5.17) on frequency Ω. In Fig. 15, we show the dispersion relation $\kappa(\Omega)$.

Comparison with Asymptotic Theory. We now compare our numerical results with the asymptotic approximation valid for small Ω. Effective medium theory predicts the leading behavior as Ω tends to zero to be

$$\kappa \sim \frac{\Omega}{c_{\text{eff}}} \tag{5.20}$$

where

$$c_{\text{eff}} = \frac{1}{\sqrt{\bar{\rho}\overline{(\lambda + 2\mu)^{-1}}}} \tag{5.21}$$

and $\bar{f}$ is the average of the function f on the interval [0, 1]. With the profiles described above, (5.21) gives $c_{\text{eff}} = 1.00005$. In the limit $\Omega \to 0$, the slope of the dispersion relation calculated by our numerical scheme approaches the value predicted by effective medium theory. The nonlinearity of the dispersion relation can be approximated by using terms of higher order in Ω

$$\lambda \sim \frac{\Omega}{c_{\text{eff}}} + d_3\Omega^3. \tag{5.22}$$

From our numerical scheme, we estimate $d_3 \approx 0.4166$. The cubic correction provides an approximation to the dispersion relation that is valid over a larger region of Ω than the linear approximation, but from Fig. 15 it is clear that for larger values of Ω, more terms in the expansion are needed. Further, an expansion based on the small-Ω limit can never

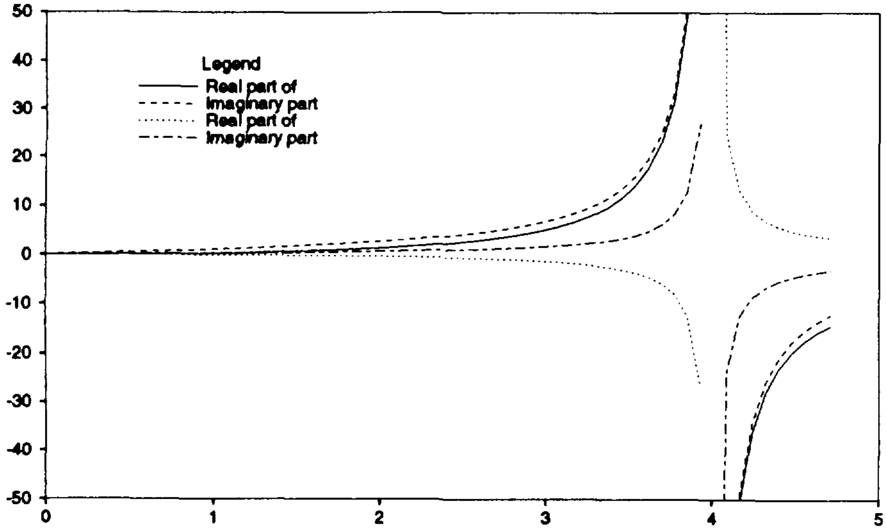

FIG. 14. Coefficients of the reduced equations as functions of frequency. The coefficients are smooth at the edge of the stopping band at ≈ 2. The coefficients have a pole at ≈ 4.

adequately describe the dispersion relation in and beyond the first stopping band.

Interpolation. Our numerical scheme is based on a calculation for a fixed value of Ω. Since for most frequencies the dispersion relation varies smoothly, we may wish to interpolate in frequency to obtain intermediate values rather than repeat the calculation for a new value of Ω. Those frequencies at the edges of stopping/passing bands are often of special interest because of their physical significance. Notice in Fig. 14 that the coefficients of the reduced scheme are smooth at the edges of the stopping/passing bands while the dispersion relation is not. We note from (5.18) and (4.10) that

$$\exp(iK) = \left(I + B_0^{(-\infty)} - \frac{1}{2}A_0^{(-\infty)}\right)^{-1} A_0^{(-\infty)}. \tag{5.23}$$

Thus we may locate the edges of the stopping/passing bands by calculating the values of Ω at which either

$$d_1 = \det\left(\left(I + B_0^{(-\infty)} - \frac{1}{2}A_0^{(-\infty)}\right)^{-1} A_0^{(-\infty)}\right) = 0 \tag{5.24}$$

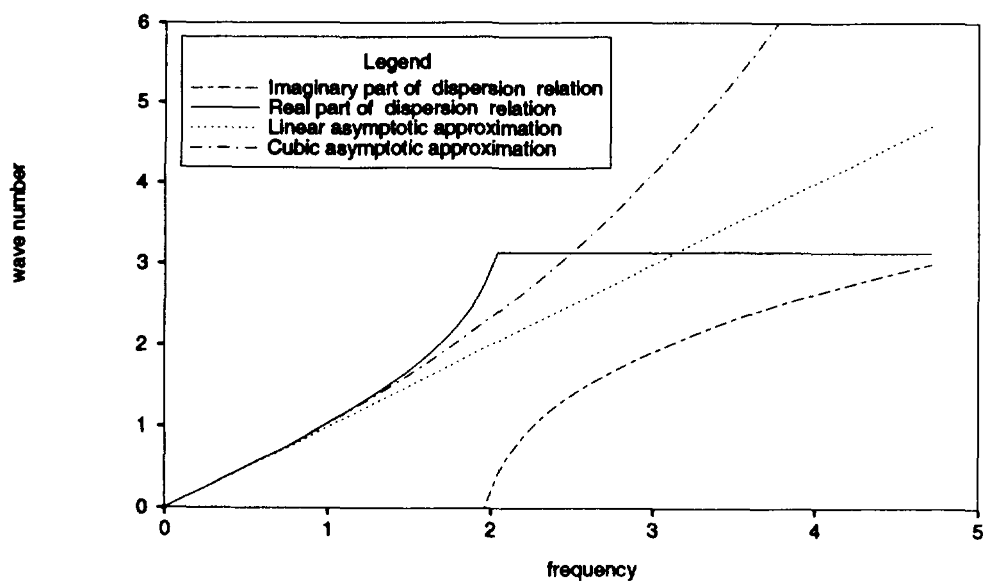

FIG. 15. Dispersion relation for 2-phase material.

or

$$d_2 = \det\left(2I + \left(I + B_0^{(-\infty)} - \frac{1}{2}A_0^{(-\infty)}\right)^{-1} A_0^{(-\infty)}\right) = 0. \quad (5.25)$$

These determinants are plotted in Fig. 16 as a function of Ω for the two-phase material described above. Note that these functions are smooth with respect to Ω and therefore it is easy to interpolate and to estimate the location of the zeros. For instance, the edge of the stopping band for the two-phase material may be calculated analytically as

$$\Omega \approx 1.99990001, \quad (5.26)$$

A numerical calculation based on spline interpolation from a set of calculations with spacing in frequency of appox. 0.08 gives the result

$$\Omega \approx 1.99996. \quad (5.27)$$

The coefficients of the reduced scheme have singularities of their own, typically simple poles located in the interior of stopping bands. In a small interval far from the poles we may obtain a good approximation of the coefficients of the reduced scheme with a spline fit and then calculate the dispersion relation from the interpolated values if desired. Rational interpolation provides a better approximation to the coefficients of the reduced scheme on intervals near or including the poles.

We have seen that the reduced scheme provides an effective means to interpolate the dispersion relation and locate stopping/passing band edges from a few calculations spread out over a large interval. The usual approach is to expand the dispersion relation in a Taylor series about the origin but this asymptotic approximation is valid only for low frequencies.

Continuous Profile. We now consider an example of a continuous but variable periodic profile. In particular we take $B = 0$ and

$$A_{12} = 1, \quad A_{21} = c_1 + c_2 \sin 2\pi x. \quad (5.28)$$

In mechanical applications, this system is referred to as the parametric oscillator (if stable) or the parametric amplifier (if unstable). Then Eq. (5.28) reduces to Mathieu's equation [4, 2]. It is well known that for $c_1 > -c_2$ there exist an infinite number of stopping bands, which in this context are called bands of instability. In the numerical calculations, the continuous profile is first approximated by its projection onto the subspace $\mathbf{V}_n$. At each level of resolution n we have a piecewise constant profile with discontinuities at intervals of 2^n. The homogenization procedure is then applied as above. In our example we set $c_1 = 2$ and $c_2 = 1$. The dispersion relation is shown in Fig. 17 for level of resolution $n = -4$. No change is observed in the graph as n is decreased beyond -4. As n tends to $-\infty$, the numerical calculations suggest the pointwise convergence of the dispersion relation to a limit.

Extensions. We have observed that the discrete formulation has coefficients which are smooth as a function of Ω near the edges of stopping/passing bands where the dispersion relation has a square-root singularity, which is an advantage for interpolation near this physically-significant

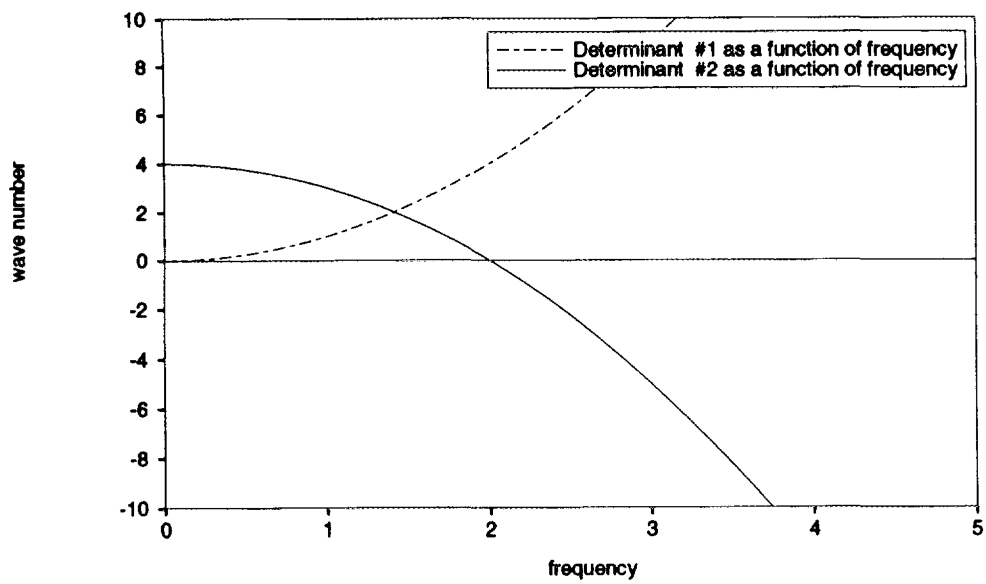

FIG. 16. Determinants for 2-phase material. Determinant d_1 vanishes at zero. Determinant d_2 vanishes at ≈ 2, the beginning of the stopping band.

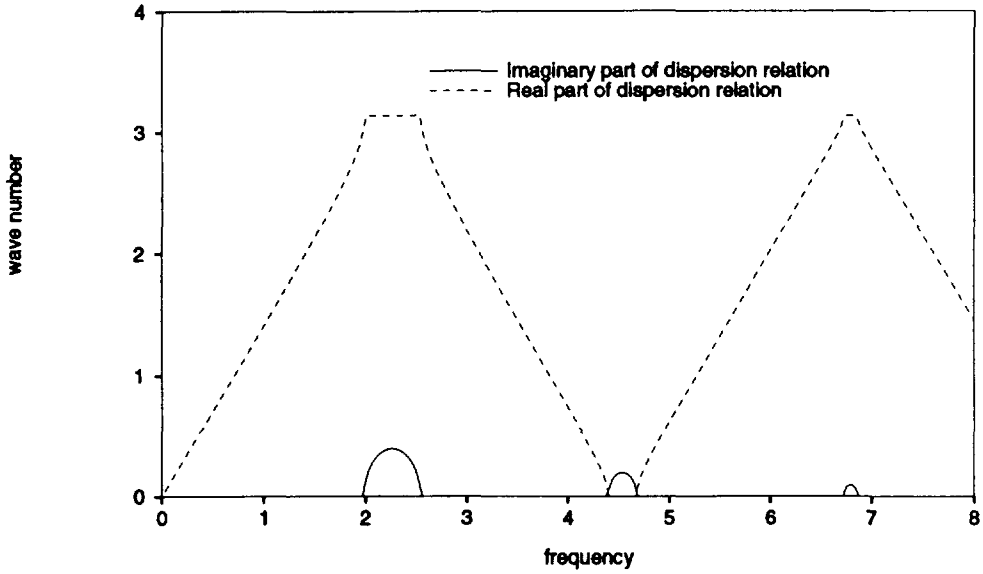

FIG. 17. Dispersion relation for Mathieu's equation.

frequency value. Another advantage of the discrete structure over a dispersion relation is the ability to easily handle aperiodic, and finite or semi-infinite media. For a given medium there is an equivalent reduced scheme (5.16) with frequency-dependent coefficients $A_{j,k}^{(-\infty)}(\Omega), B_{j,k}^{(-\infty)}(\Omega)$ such that the solution of (3.32) gives the projection on $\mathbf{V}_j$ of the solution of the original equation (5.13). If the medium is aperiodic it may not be possible to describe this propagation in terms of a dispersion relation however. The reduced scheme provides a coarse-scale formulation that does not assume periodicity. Also boundary conditions for a finite or semi-infinite medium are easily incorporated into the reduced scheme.

6. APPENDIX A

Proof of Proposition II.1. We proceed by induction. Assume that Eq. (2.11) has been obtained for multiresolution level $j-1$,

$$\mathbf{B}_{j-1}^{(n)} x_{j-1}^{(n)} + q_{j-1}^{(n)} + \lambda = \mathbf{K}_{j-1}(\mathbf{A}_{j-1}^{(n)} x_{j-1}^{(n)} + p_{j-1}^{(n)}). \tag{6.1}$$

Let us modify (6.1) using projections onto the spaces $\mathbf{V}_j$ and $\mathbf{W}_j$ to derive the recursion relations for $\mathbf{A}_j^{(n)}, \mathbf{B}_j^{(n)}, q_j^{(n)}, p_j^{(n)}$ in terms of $\mathbf{A}_{j-1}^{(n)}, \mathbf{B}_{j-1}^{(n)}, q_{j-1}^{(n)}, p_{j-1}^{(n)}$.

Let us rewrite Eq. (6.1) in terms of unknowns in the subspaces $\mathbf{V}_j$ and $\mathbf{W}_j$. We start by noting that since $x_{j-1}^{(n)} \in \mathbf{V}_{j-1}$, it may be represented as

$$x_{j-1}^{(n)} = \mathbf{P}_j x_{j-1}^{(n)} + \mathbf{Q}_j x_{j-1}^{(n)}, \tag{6.2}$$

so that we write

$$u \equiv \mathbf{P}_j x^{(n)} = x_j^{(n)} \quad \text{and} \quad \frac{1}{\epsilon_j} v \equiv \mathbf{Q}_j x^{(n)}, \tag{6.3}$$

where ϵ_j is a scaling factor that we will choose appropriately for a given MRA. In general, ϵ_j will represent a typical order of magnitude of $\mathbf{Q}_j x^{(n)}$ so that v will be $O(1)$. Using (6.2) and (6.3), we have

$$x_{j-1}^{(n)} = u + \epsilon_j v, \tag{6.4}$$

and substituting (6.4) into (6.1), we obtain

$$\mathbf{B}_{j-1}^{(n)}(u+\epsilon_j v) + q_{j-1}^{(n)} + \lambda = \mathbf{K}_{j-1}(\mathbf{A}_{j-1}^{(n)}(u+\epsilon_j v) + p_{j-1}^{(n)}). \tag{6.5}$$

Further, we split (6.5) into two equations by applying $\mathbf{P}_j$ and $\mathbf{Q}_j$ so that

$$\mathbf{P}_j\mathbf{B}_{j-1}^{(n)}(u + \epsilon_j v) + \mathbf{P}_j q_{j-1}^{(n)} + \lambda = \mathbf{P}_j\mathbf{K}_{j-1}(\mathbf{A}_{j-1}^{(n)}(u + \epsilon_j v) + p_{j-1}^{(n)}), \tag{6.6}$$

$$\mathbf{Q}_j\mathbf{B}_{j-1}^{(n)}(u + \epsilon_j v) + \mathbf{Q}_j q_{j-1}^{(n)} = \mathbf{Q}_j\mathbf{K}_{j-1}(\mathbf{A}_{j-1}^{(n)}(u + \epsilon_j v) + p_{j-1}^{(n)}). \tag{6.7}$$

Since $u \in \mathbf{V}_j$ and $v \in \mathbf{W}_j$, applying the corresponding projection operators $\mathbf{P}_j$ and $\mathbf{Q}_j$ will leave these elements unchanged. Thus we have

$$u = \mathbf{P}_j u \quad \text{and} \quad v = \mathbf{Q}_j v. \tag{6.8}$$

Therefore, we have

$$
\begin{aligned}
\mathbf{P}_j\mathbf{B}_{j-1}^{(n)}u &= \mathbf{P}_j\mathbf{B}_{j-1}^{(n)}\mathbf{P}_j u = \mathcal{T}_{\mathbf{B},j}^{(n)}u \\
\mathbf{P}_j\mathbf{B}_{j-1}^{(n)}v &= \mathbf{P}_j\mathbf{B}_{j-1}^{(n)}\mathbf{Q}_j v = \epsilon_j\mathcal{C}_{\mathbf{B},j}^{(n)}v \\
\mathbf{Q}_j\mathbf{B}_{j-1}^{(n)}u &= \mathbf{Q}_j\mathbf{B}_{j-1}^{(n)}\mathbf{P}_j u = \epsilon_j\mathcal{B}_{\mathbf{B},j}^{(n)}u \\
\mathbf{Q}_j\mathbf{B}_{j-1}^{(n)}v &= \mathbf{Q}_j\mathbf{B}_{j-1}^{(n)}\mathbf{Q}_j v = \mathcal{A}_{\mathbf{B},j}^{(n)}v.
\end{aligned}
\tag{6.9}
$$

and, similarly,

$$
\begin{aligned}
\mathbf{P}_j\mathbf{K}_{j-1}\mathbf{A}_{j-1}^{(n)}u &= \mathbf{P}_j\mathbf{K}_{j-1}\mathbf{P}_j\mathbf{P}_j\mathbf{A}_{j-1}^{(n)}\mathbf{P}_j u \\
&\quad + \mathbf{P}_j\mathbf{K}_{j-1}\mathbf{Q}_j\mathbf{Q}_j\mathbf{A}_{j-1}^{(n)}\mathbf{P}_j u \\
&= \mathcal{T}_{\mathbf{K},j}\mathcal{T}_{\mathbf{A},j}^{(n)}u + \epsilon_j^2\mathcal{C}_{\mathbf{K},j}\mathcal{B}_{\mathbf{A},j}^{(n)}u \\
\mathbf{P}_j\mathbf{K}_{j-1}\mathbf{A}_{j-1}^{(n)}v &= \mathbf{P}_j\mathbf{K}_{j-1}\mathbf{P}_j\mathbf{P}_j\mathbf{A}_{j-1}^{(n)}\mathbf{Q}_j v \\
&\quad + \mathbf{P}_j\mathbf{K}_{j-1}\mathbf{Q}_j\mathbf{Q}_j\mathbf{A}_{j-1}^{(n)}\mathbf{Q}_j v \\
&= \epsilon_j\mathcal{T}_{\mathbf{K},j}\mathcal{C}_{\mathbf{A},j}^{(n)}v + \epsilon_j\mathcal{C}_{\mathbf{K},j}\mathcal{A}_{\mathbf{A},j}^{(n)}v \qquad (6.10)\\
\mathbf{Q}_j\mathbf{K}_{j-1}\mathbf{A}_{j-1}^{(n)}u &= \mathbf{Q}_j\mathbf{K}_{j-1}\mathbf{P}_j\mathbf{P}_j\mathbf{A}_{j-1}^{(n)}\mathbf{P}_j u \\
&\quad + \mathbf{Q}_j\mathbf{K}_{j-1}\mathbf{Q}_j\mathbf{Q}_j\mathbf{A}_{j-1}^{(n)}\mathbf{P}_j u \\
&= \epsilon_j\mathcal{B}_{\mathbf{K},j}\mathcal{T}_{\mathbf{A},j}^{(n)}u + \epsilon_j^3\mathcal{A}_{\mathbf{K},j}\mathcal{B}_{\mathbf{A},j}^{(n)}u \\
\mathbf{Q}_j\mathbf{K}_{j-1}\mathbf{A}_{j-1}^{(n)}v &= \mathbf{Q}_j\mathbf{K}_{j-1}\mathbf{P}_j\mathbf{P}_j\mathbf{A}_{j-1}^{(n)}\mathbf{Q}_j v \\
&\quad + \mathbf{Q}_j\mathbf{K}_{j-1}\mathbf{Q}_j\mathbf{Q}_j\mathbf{A}_{j-1}^{(n)}\mathbf{Q}_j v \\
&= \epsilon_j^2\mathcal{B}_{\mathbf{K},j}\mathcal{C}_{\mathbf{A},j}^{(n)}v + \epsilon_j^2\mathcal{A}_{\mathbf{K},j}\mathcal{A}_{\mathbf{A},j}^{(n)}v.
\end{aligned}
$$

We also expand the forcing terms into their projections onto $\mathbf{V}_j$ and $\mathbf{W}_j$. Let

$$
S_{q,j}^{(n)} = \mathbf{P}_j q_{j-1}^{(n)}, \quad S_{p,j}^{(n)} = \mathbf{P}_j p_{j-1}^{(n)}, \tag{6.11}
$$

and

$$
D_{q,j}^{(n)} = \frac{1}{\epsilon_j}\mathbf{Q}_j q_{j-1}^{(n)}, \quad D_{p,j}^{(n)} = \frac{1}{\epsilon_j}\mathbf{Q}_j p_{j-1}^{(n)}. \tag{6.12}
$$

Then we obtain

$$
\begin{aligned}
\mathbf{P}_j\mathbf{K}_{j-1}p_j &= \mathbf{P}_j\mathbf{K}_{j-1}(S_{p,j}^{(n)} + \epsilon_j D_{p,j}^{(n)}) \\
&= \mathbf{P}_j\mathbf{K}_{j-1}\mathbf{P}_j S_{p,j}^{(n)} + \epsilon_j\mathbf{P}_j\mathbf{K}_{j-1}\mathbf{Q}_j D_{p,j}^{(n)} \\
&= \mathcal{T}_{\mathbf{K},j}S_{p,j}^{(n)} + \epsilon_j^2\mathcal{C}_{\mathbf{K},j}D_{p,j}^{(n)}, \\
\mathbf{Q}_j\mathbf{K}_{j-1}p_j &= \mathbf{P}_j\mathbf{K}_{j-1}(S_{p,j}^{(n)} + \epsilon_j D_{p,j}^{(n)}) \\
&= \mathbf{Q}_j\mathbf{K}_{j-1}\mathbf{P}_j S_{p,j}^{(n)} + \epsilon_j\mathbf{P}_j\mathbf{K}_{j-1}\mathbf{Q}_j D_{p,j}^{(n)} \\
&= \epsilon_j\mathcal{B}_{\mathbf{K},j}S_{p,j}^{(n)} + \epsilon_j^3\mathcal{A}_{\mathbf{K},j}D_{p,j}^{(n)}.
\end{aligned}
\tag{6.13}
$$

Substituting (6.9)–(6.14) into Eqs. (6.6), (6.7), we arrive at

$$
\begin{aligned}
&(\mathcal{T}_{\mathbf{B},j}^{(n)} - \epsilon_j^2\mathcal{C}_{\mathbf{K},j}\mathcal{C}_{\mathbf{A},j}^{(n)})u + \epsilon_j^2(\mathcal{B}_{\mathbf{B},j}^{(n)} - \mathcal{C}_{\mathbf{K},j}\mathcal{A}_{\mathbf{A},j}^{(n)})v \\
&\quad + S_{q,j}^{(n)} - \epsilon_j^2\mathcal{C}_{\mathbf{K},j}D_{p,j}^{(n)} + \lambda \\
&\qquad = \mathcal{T}_{\mathbf{K},j}(\mathcal{T}_{\mathbf{A},j}^{(n)}u + \epsilon_j^2\mathcal{C}_{\mathbf{A},j}^{(n)}v + S_{p,j}^{(n)}) \\
&(\mathcal{A}_{\mathbf{B},j}^{(n)} - \epsilon_j^2\mathcal{B}_{\mathbf{K},j}\mathcal{C}_{\mathbf{A},j}^{(n)} - \epsilon_j^2\mathcal{A}_{\mathbf{K},j}\mathcal{A}_{\mathbf{A},j}^{(n)})v \\
&\quad = -D_{q,j}^{(n)} + (\mathcal{B}_{\mathbf{K},j}\mathcal{T}_{\mathbf{A},j}^{(n)} + \epsilon_j^2\mathcal{A}_{\mathbf{K},j}\mathcal{B}_{\mathbf{A},j}^{(n)} - \mathcal{B}_{\mathbf{B},j}^{(n)})u \\
&\qquad + \mathcal{B}_{\mathbf{K},j}S_{p,j}^{(n)} + \epsilon_j^2\mathcal{A}_{\mathbf{K},j}D_{p,j}^{(n)}.
\end{aligned}
\tag{6.14}
$$

Equations (6.15) are equivalent to the original equation in $\mathbf{V}_{j-1}$ (6.1). If the operator $\mathbf{F}_j^{(n)}$,

$$
\mathbf{F}_j^{(n)} = \mathcal{A}_{\mathbf{B},j}^{(n)} - \epsilon_j^2\mathcal{B}_{\mathbf{K},j}\mathcal{C}_{\mathbf{A},j}^{(n)} - \epsilon_j^2\mathcal{A}_{\mathbf{K},j}\mathcal{A}_{\mathbf{A},j}^{(n)} \tag{6.15}
$$

is invertible, then we solve the second equation of system (6.14) for v,

$$
v = -\mathbf{C}_j^{(n)}u - r_j^{(n)} \tag{6.16}
$$

where

$$
\mathbf{C}_j^{(n)} = \mathbf{F}_j^{(n)^{-1}}(\mathcal{B}_{\mathbf{B},j}^{(n)} - \mathcal{B}_{\mathbf{K},j}\mathcal{T}_{\mathbf{A},j}^{(n)} - \epsilon_j^2\mathcal{A}_{\mathbf{K},j}\mathcal{B}_{\mathbf{A},j}^{(n)}), \tag{6.17}
$$

$$
r_j^{(n)} = \mathbf{F}_j^{(n)^{-1}}(D_{q,j}^{(n)} - \mathcal{B}_{\mathbf{K},j}S_{p,j}^{(n)} - \epsilon_j^2\mathcal{A}_{\mathbf{K},j}D_{p,j}^{(n)}). \tag{6.18}
$$

7. APPENDIX B

Proof of Proposition II.2. From the properties of the MRA we have that the sequences $\mathbf{A}_n$, $\mathbf{B}_n$, and $\mathbf{K}_n$ converge to $\mathbf{A}, \mathbf{B}$, and $\mathbf{K}$ as n tends to $-\infty$. Also because $\mathbf{P}_n$ is an orthogonal projection, we have

$$
\|\mathbf{P}_n\| \leq 1, \tag{7.1}
$$

which, in turn, implies

$$
\|\mathbf{A}_n\| \leq \|\mathbf{A}\|. \tag{7.2}
$$

The same holds for sequences of operators $\mathbf{B}_n$ and $\mathbf{K}_n$. Therefore there is some $n_0 \leq 0$ such that for all $n \leq n_0$ we have

$$
\begin{aligned}
\|(\mathbf{B}_n - \mathbf{K}_n\mathbf{A}_n) - (\mathbf{B} - \mathbf{K}\mathbf{A})\| &\leq \|\mathbf{B} - \mathbf{B}_n\| + \|\mathbf{K}\|\,\|\mathbf{A} - \mathbf{A}_n\| \\
&\quad + \|\mathbf{A}\|\,\|\mathbf{K} - \mathbf{K}_n\| \leq \frac{1}{2\|(\mathbf{B} - \mathbf{K}\mathbf{A})^{-1}\|}.
\end{aligned}
\tag{7.3}
$$

We now use the following inequality which follows from the Cauchy–Schwartz inequality: if K is an invertible operator and

$$
\|M - K\| < \frac{1}{\|K^{-1}\|}, \tag{7.4}
$$

then M is also invertible and

$$\|M^{-1}\| \leqslant \frac{\|K^{-1}\|}{1 - \|M - K\| \|K^{-1}\|}. \tag{7.5}$$

Then from (7.3) we have that $\mathbf{B}_n - \mathbf{K}_n\mathbf{A}_n$ is invertible for $n \leqslant n_0 \leqslant 0$. Further, the inverses of these operators are uniformly bounded. In particular,

$$\|(\mathbf{B}_n - \mathbf{K}_n\mathbf{A}_n)^{-1}\| \leqslant 2\|(\mathbf{B} - \mathbf{KA})^{-1}\|. \tag{7.6}$$

Next we establish the convergence of the sequence $x^{(n)}$ to the solutions x. From (2.1) and (2.8) we have

$$\begin{aligned}\|x - x^{(n)}\| &\leqslant \|(\mathbf{B}_n - \mathbf{K}_n\mathbf{A}_n)^{-1}(\mathbf{K}_n p_n - q_n) \\ &\quad - (\mathbf{B} - \mathbf{KA})^{-1}(\mathbf{K}p - q)\| \\ &\leqslant \|\mathbf{B} - \mathbf{KA}^{-1}\|(\|\mathbf{K}p - \mathbf{K}_n p_n\| \\ &\quad + \|q - q_n\|) + \|\mathbf{K}_n p_n - q_n\| \|(\mathbf{B} - \mathbf{KA})^{-1} \\ &\quad - (\mathbf{B}_n - \mathbf{K}_n\mathbf{A}_n)^{-1}\|. \end{aligned} \tag{7.7}$$

Now let

$$C_1 = \|(\mathbf{B} - \mathbf{KA})^{-1}\| \tag{7.8}$$

and

$$C_2 = \|\mathbf{K} - q\|. \tag{7.9}$$

Then

$$\begin{aligned}\|x - x^{(n)}\| &\leqslant C_1(\|\mathbf{K}_p - \mathbf{K}_n p_n\| + \|q - q_n\|) \\ &\quad + C_2\|(\mathbf{B} - \mathbf{KA})^{-1}\|(\|\mathbf{B} - \mathbf{B}_n\| \\ &\quad + \|\mathbf{KA} - \mathbf{K}_n\mathbf{A}_n\|)\|(\mathbf{B}_n - \mathbf{K}_n\mathbf{A}_n)^{-1}\| \\ &\leqslant C_1(\|\mathbf{K}\| \|p - p_n\| + \|\mathbf{K} - \mathbf{K}_n\| \|p\| \\ &\quad + \|q - q_n\|) + 2C_2C_1^2 \\ &\quad \times (\|\mathbf{B} - \mathbf{B}_n\| + \|\mathbf{K}\| \|\mathbf{A} - \mathbf{A}_n\| \\ &\quad + \|\mathbf{K} - \mathbf{K}_n\| \|\mathbf{A}\|) \leqslant C_3\|I - \mathbf{P}_n\|, \end{aligned} \tag{7.10}$$

where

$$C_3 = C_1(2\|\mathbf{K}\| \|p\| + \|q\|) + 2C_1^2C_2(\|\mathbf{B}\| + 2\|\mathbf{K}\| \|\mathbf{A}\|). \tag{7.11}$$

Now since $\|I - \mathbf{P}_n\|$ vanishes as n tends to $-\infty$, we have established that $x^{(n)}$ tends to x as n tends to $-\infty$.

Proof of Proposition II.3. To establish (2.36), we find an upper bound on the residual and show that it vanishes. Let j be fixed, and let n be an element of the subsequence for which the limits (2.35) exist. From (2.25) we have

$$\begin{aligned}&\|\mathbf{B}_j^{(-\infty)}x_j + q_j^{(-\infty)} - \mathbf{K}_j(\mathbf{A}_j^{(-\infty)}x_j - p_j^{(-\infty)})\| \\ &= \|(\mathbf{B}_j^{(-\infty)}x_j + q_j^{(-\infty)} - \mathbf{K}_j(\mathbf{A}_j^{(-\infty)}x_j - p_j^{(-\infty)})) \\ &\quad - (\mathbf{B}_j^{(n)}x_j^{(n)} + q_j^{(n)} - \mathbf{K}_j(\mathbf{A}_j^{(n)}x_j^{(n)} - p_j^{(n)}))\|. \end{aligned} \tag{7.12}$$

Using Cauchy-Schwarz and triangle inequalities we find

$$\begin{aligned}&\|\mathbf{B}_j^{(-\infty)}x_j + q_j^{(-\infty)} - \mathbf{K}_j(\mathbf{A}_j^{(-\infty)}x_j - p_j^{(-\infty)})\| \\ &\leqslant \|\mathbf{B}_j^{(-\infty)} - \mathbf{B}_j^{(n)}\| \|x_j^{(n)}\| + \|\mathbf{B}_j^{(-\infty)}\| \|x_j - x_j^{(n)}\| \\ &\quad + \|q_j^{(-\infty)} - q_j^{(n)}\| + \|\mathbf{K}_j\|(\|\mathbf{A}_j^{(-\infty)} - \mathbf{A}_j^{(n)}\| \|x_j^{(n)}\| \\ &\quad + \|\mathbf{A}_j^{(-\infty)}\| \|x_j - x_j^{(n)}\| + \|p_j^{(-\infty)} - p_j^{(n)}\|). \end{aligned} \tag{7.13}$$

As $n \to -\infty$ within the convergent subsequence, all terms on the right-hand side of (7.13) vanish. Hence (2.36) is established.

Note that in (2.1) we may choose $p = 0$ and q to be an arbitrary element of $\mathbf{V}_j$. Thus the range of $\mathbf{B}_j^{(-\infty)} - \mathbf{K}_j\mathbf{A}_j^{(-\infty)}$ is $\mathbf{V}_j$, since $p_j^{(-\infty)} = 0$ and $q_j^{(-\infty)} = q$. We now employ the fact the $\mathbf{P}_j$ is an orthogonal projection. For an arbitrary $q \in \mathbf{V}_j$, we have

$$\begin{aligned}&\|(\mathbf{B}_j^{(-\infty)} - \mathbf{K}_j\mathbf{A}_j^{(-\infty)})^{-1}q\| \\ &= \|x_j\| \leqslant \|x\| \leqslant \|(\mathbf{B} - \mathbf{KA})^{-1}\| \|q\|. \end{aligned} \tag{7.14}$$

Hence, (2.37) is established.

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