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Three-dimensional inverse scattering for the wave equation with variable speed: near-field formulae using point sources

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Abstract. We consider the inverse scattering problem for the wave equation with variable speed where the region of interest is probed with waves emanating from point sources. We obtain a three-dimensional trace-type formula, which gives the unknown speed in terms of data and the interior wavefield.

1. Introduction

In this paper we consider the inverse scattering problem for the wave equation with variable speed and incident fields generated by point sources. The paper follows the lines of Rose and Cheney [1], in which the incident fields were plane waves; see also [2–4].

Two special features of our work are the following. First, we obtain a theory which applies when the wavefield is known on a surface surrounding the scatterer, this surface possibly being near the scatterer. Second, the theory simplifies because the waves we consider have convenient support properties.

This paper is organised as follows. In § 2 we derive an inverse scattering equation that relates the data to the wavefield. In § 3 we derive a relation (in the zero-frequency limit) relating the unknown speed to the wavefield. We then discuss support properties and use them to derive a three-dimensional trace-type formula, which expresses the unknown speed in terms of data and the interior wavefield.

2. The inverse scattering equation

We consider the reduced wave equation with a point source:

$$[\nabla^2 + k^2 n^2(x)]G(k, x, y) = \delta(x - y). \tag{2.1}$$

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Here x and y are points in $\boldsymbol{R}^3$, k is a real scalar, and δ is the three-dimensional delta function. The index of refraction $n(x)$ we assume to be a positive, bounded, real-valued function which is identically one outside some bounded region Ω.

We are interested in particular solutions of (2.1) which we specify with the help of the functions

$$G_0^{\pm}(k,x) = -(4\pi|x|)^{-1}\exp(\pm \mathrm{i}k|x|)$$

which satisfy

$$(\nabla^2 + k^2)G_0^{\pm}(k,x-y) = \delta(x-y). \tag{2.2$\pm$}$$

We now specify solutions G^+ and G^- of (2.1) as solutions of the integral equations

$$G^{\pm}(k,x,y) = G_0^{\pm}(k,x-y) + \int_\Omega G_0^{\pm}(k,x-z)k^2V(z)G^{\pm}(k,z,y)\,\mathrm{d}z \tag{2.3$\pm$}$$

where $V = 1 - n^2$.

There are two techniques for showing that (2.3+) and (2.3−) each have unique solutions. One technique [5] shows that for almost every k, (2.3) has a unique solution with $G|V|^{1/2}$ in L^2. Another technique [6], which uses the limiting absorption principle, shows that for every k, (2.3) has a unique solution in a certain weighted Sobolev space. Both these techniques apply in the present case when V is bounded and has compact support.

Two relations following from (2.3) will be needed in § 4: first, that G^- is the complex conjugate of G^+, and second

$$G^+(-k,x,y) = G^-(k,x,y). \tag{2.4}$$

The k dependence of G is not needed for the remainder of §2; hence we will suppress k until §3. We note the symmetry of the Green function $G(x,y) = G(y,x)$ (see, e.g., [5]).

The inverse scattering problem that we consider is to determine $V(x)$ from scattering data. The particular data we use are measurements of $G^+(x,y)$ for x, y on $\partial\Omega$ and for all k. This corresponds to putting point sources and receivers on $\partial\Omega$. If $\partial\Omega$ is close to the support of V, the data contain 'near-field' information.

The following theorem gives an 'inverse scattering equation' for this problem. It is the point-source analogue of an equation in [3].

Theorem 1. Suppose $n^2(x)$ is positive, real-valued, and has two continuous derivatives. Assume $n^2(x) = 1$ outside Ω and that $\partial\Omega$ is smooth. Then

$$\int_{\partial\Omega}\left(G^-(z,x)\frac{\partial}{\partial\nu}G^+(z,y) - G^+(z,y)\frac{\partial}{\partial\nu}G^-(z,x)\right)\mathrm{d}S_z = G^-(y,x) - G^+(x,y) \tag{2.5}$$

where ν denotes the outward unit normal to $\partial\Omega$.

Proof. The proof, based on the use of Green's formula, is similar to the corresponding proof in [3] and is omitted here.

Corollary 1.

$$\int_{\partial\Omega}\left(G_0^-(z-x)\frac{\partial}{\partial\nu}G_0^+(z-y)-G_0^+(z-y)\frac{\partial}{\partial\nu}G_0^-(z-x)\right)\mathrm{d}S_z$$
$$=G_0^-(y-x)-G_0^+(x-y). \tag{2.6}$$

Proof. We set $n(x)$ equal to one in theorem 1.

Corollary 2. Suppose the hypotheses of theorem 1 hold and $G^{\pm}=G_0^{\pm}+G_{\mathrm{sc}}^{\pm}$. Then

$$G_{\mathrm{sc}}^-(y,x)-G_{\mathrm{sc}}^+(x,y)=\int_{\partial\Omega}\left(G_0^-(z-x)\frac{\partial}{\partial\nu}G_{\mathrm{sc}}^+(z,y)-G_{\mathrm{sc}}^+(z,y)\frac{\partial}{\partial\nu}G_0^-(z-x)\right.$$
$$\left.+G_{\mathrm{sc}}^-(z,x)\frac{\partial}{\partial\nu}G^+(z,y)-G^+(z,y)\frac{\partial}{\partial\nu}G_{\mathrm{sc}}^-(z,x)\right)\mathrm{d}S_z. \tag{2.7}$$

Proof. We merely subtract (2.6) from (2.5).

Corollaries 1 and 2 hold also in the case when instead of taking our 'reference' index of refraction to be identically one, we take it to be some $n_0(x)$. We denote the corresponding solutions of (2.1) by $\tilde{G}_0^{\pm}$:

$$(\nabla^2+k^2n_0^2(x))\tilde{G}_0^{\pm}(x,y)=\delta(x-y). \tag{2.8}$$

If $n_0(x)$ also satisfies the hypotheses of theorem 1, then (2.6) and (2.7) hold when the $G_0^{\pm}$ are replaced by the $\tilde{G}_0^{\pm}$. This formulation may be especially useful when n is a small perturbation of n_0; however, this is not pursued in the present paper.

Both (2.5) and (2.7) can be considered inverse scattering equations in the following sense. Suppose we fix y on $\partial\Omega$. Then the terms of (2.5) and (2.7) involving $G^+(z, y)$ can be considered to be data. They correspond to a point source at y and receiver at z on $\partial\Omega$. Equations (2.5) and (2.7) thus relate the wavefields G^+ and G^- at a point x in Ω to the data measured on $\partial\Omega$.

3. The trace-type formula

This section follows [3]. We first derive a relation between V and the zero-frequency wavefield. This relation is then used together with (2.7) and time-domain information to obtain formula (3.7) for V in terms of the scattering data and the interior wavefield.

We begin with the relation between V and the zero-frequency wavefield. This relation, in a slightly different form, appears in [7]. It shows how to find V from knowledge of $G^+(k, x, y)$ for some fixed y, all x, and k near zero.

Lemma: Suppose $V(x)=1-n^2(x)$ is in L^2 and L^1. Then

$$V(x)=-4\pi|x-y|\nabla^2[k^{-2}G_{\mathrm{sc}}^+(k,x,y)]_{k=0}. \tag{3.1}$$

Proof: We substitute $G^+ = G_0^+ + G_{sc}^+$ into (2.1), simplify, and evaluate at $k=0$.

Next, we need some time-domain information. Accordingly, we consider the distributional Fourier transforms of G^+ and G^-:

$$g^{\pm}(t, x, y) = (2\pi)^{-1} \int_{-\infty}^{\infty} G^{\pm}(k, x, y) \exp(-\mathrm{i}kt)\, \mathrm{d}k. \tag{3.2}$$

The precise sense in which (3.2) exists is discussed in [8]. The functions $g^{\pm}$ satisfy the wave equation

$$(\nabla^2 - n^2(x)\partial_{tt})g^{\pm}(t, x, y) = \delta(x-y)\delta(t). \tag{3.3}$$

We note that (2.4) implies

$$g^+(-t, x, y) = g^-(t, x, y). \tag{3.4}$$

The time domain is needed to understand certain facts about the supports of g^+ and g^- which derive from the finiteness of $n^{-1}(x)$, the speed of propagation in (3.3). For each x, there is some time at which the signal g_{sc} from y first reaches x. We denote this time by $s(x, y)$. Thus we have

$$g_{sc}^+(t, x, y) = 0 \qquad \text{for } t < s(x, y). \tag{3.5}$$

From (3.4), we have also

$$g_{sc}^-(t, x, y) = 0 \qquad \text{for } t > -s(x, y). \tag{3.6}$$

We note that since $s(x, y) \geqslant 0$, the supports of g^+ and g^- intersect at $t=0$, $x=y$ only.

We are now in a position to state and prove the following trace-type formula.

Theorem 2: Suppose $n^2(x)$ is positive, real-valued, and has two continuous derivatives. Assume $n^2(x)=1$ outside Ω, and that $\partial\Omega$ is smooth. Then $V(x) = 1 - n^2(x)$ is given by

$$V(x) = -2\mathrm{i}|x-y|\nabla^2 \int_{-\infty}^{\infty} \exp(-\mathrm{i}k\sigma(x, y))\,(k+\mathrm{i}0)^{-3} D(k, x, y)\, \mathrm{d}k \tag{3.7}$$

where σ is any smooth function satisfying $-s(x, y) < \sigma(x, y) < s(x, y)$, and where

$$D(k, x, y) = \int_{\partial\Omega} \Bigg[G_0^-(k, z-y) \frac{\partial}{\partial\nu} G_{sc}^+(k, z, x) - G_{sc}^+(k, z, x) \frac{\partial}{\partial\nu} G_0^-(k, z-y)$$
$$+ G_{sc}^-(k, z, y) \frac{\partial}{\partial\nu} G^+(k, z, x) - G^+(k, z, x) \frac{\partial}{\partial\nu} G_{sc}^-(k, z, y) \Bigg] \mathrm{d}S_z.$$

Formula (3.7) we call the trace-type formula, because it has a structure similar to that of the Deift–Trubowitz one-dimensional formula [9]. In particular, formula (3.7) expresses V in terms of the interior wavefield and the data (which in this case are measurements of the wavefield on the boundary $\partial\Omega$). Formula (3.7) is different from the trace formula [9], however, in that it is linear rather than quadratic in the wavefield and in that it contains derivatives.

Proof of theorem 2. Following [1] we apply support information to the computation of V from G (3.1). We denote by h^+ the (distributional) Fourier transform of G^+_{sc}/k^2:

$$k^{-2}G^+_{sc}(k, x, y) = \int_{-\infty}^{\infty} h^+(t, x, y) \exp(ikt)\, dt. \tag{3.8}$$

Since h^+ is the second antiderivative of g^+, it satisfies the same causality property (3.5). Thus the lower limit of integration in (3.8), may be taken to be σ where $-s<\sigma<s$. Equation (3.1) can therefore be written

$$V(x) = -4\pi|x-y|\nabla^2 \int_{\sigma(x,y)}^{\infty} h^+(t, x, y)\, dt. \tag{3.9}$$

Next we compute h^+ with the help of (2.7). We divide both sides of (2.7) by k^2 and then apply the Fourier transform to obtain

$$h^+(t, x, y) - h^-(t, x, y) = -(2\pi)^{-1} \int_{-\infty}^{\infty} k^{-2} \exp(-ikt) D(k, x, y)\, dk. \tag{3.10}$$

We then integrate (3.10) in t from $\sigma(x, y)$ to infinity and substitute the result into the right-hand side of (3.9). This gives the following expression for V:

$$V(x) = 2|x-y|\nabla^2 \int_{\sigma(x,y)}^{\infty} \int_{-\infty}^{\infty} k^{-2} \exp(-ikt) D(k, x, y)\, dk\, dt. \tag{3.11}$$

Finally, we carry out the integration with respect to t in (3.11). To accomplish this we change variables to $t' = t - \sigma(x, y)$ and use the formula (e.g., [10]) for the Fourier transform of the Heaviside function $H(t)$:

$$\int_{-\infty}^{\infty} H(t) \exp(-ikt)\, dt = \text{PV}(ik)^{-1} + \pi\delta(k) = [i(k+i0)]^{-1}. \tag{3.12}$$

Remarks 1. Since D is zero at $k=0$, the delta term of (3.12) does not contribute to (3.7).

Remark 2. An interesting feature of (3.7) is the arbitrariness of σ. A similar arbitrariness arises in [11] and is found to be very useful.

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Refereces

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