Inverse Problems 3 (1987) 683–690. Printed in the UK

Wavefront sets of solutions to linearised inverse scattering problems

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Received 23 February 1987

Abstract. A certain integral operator T, which arises in some linearised inverse scattering problems, is shown to be a pseudodifferential operator and its points of ellipticity are explicitly described. Consequently for the equation $Tf=g$, one can find the relationship between the wavefront sets of f and g. From this viewpoint a class of algorithms called the migration schemes and used extensively in geophysics for imaging the Earth's interior can be understood clearly.

1. Introduction

The classical notion of a solution to an integral or differential equation refers to a function that satisfies the equation exactly or with a small discrepancy. The notion of a parametrix which is central in the theory of pseudodifferential and Fourier integral operators breaks with this tradition and allows the discrepancy to be any smooth function. This point of view has proved to be very fruitful in studying the properties of solutions of various partial differential and integral equations [1–9], and recently found its place in applications [10–13].

In problems of non-destructive evaluation, many practical questions can be adequately answered provided we can accurately reconstruct discontinuities in the parameters of the physical medium. Seismic exploration, medical applications, crack and void detection are examples of the variety of fields where these problems are of interest.

In seismic exploration a class of various algorithms developed semiheuristically over the years (e.g. [14–22]) and aimed at recovering essentially the discontinuities of the wave velocity are called migration algorithms. These algorithms can be viewed as approximate solutions to a linearised inverse scattering problem. By 'an inverse scattering problem' we mean one that involves reconstructing the wave velocity (or other parameters) of the medium from measurements of the scattered field over a surface of a codimension one in space. More explicitly, a migration algorithm computes a function $g(x)$ on space from these scattered data along the surface. As one often attempts to recover the discontinuities of the true wave velocity from those of g, the following two principal issues are of concern.

(i) Are the discontinuities of g part of the discontinuities of the true wave veclocity?

(ii) To what extent are the discontinuities of the true wave velocity preserved as the discontinuities of g?

0266-5611/87/040683 + 08 \$02.50

In [10] precise answers to these questions are given using the notion of a generalised Radon transform [23]. In particular the answer is affirmative to the first question.

The purpose of this short paper is to reformulate the results of [10] in terms of wavefront sets. This provides a natural framework for formulating the inverse problem in general as one of determination of wavefront sets of unknown parameters of PDEs, which is a meaningful question to ask from the point of view of applications.

Such a formulation is also meaningful from the pure mathematical point of view. In other words, the wavefront set of a function seems to be a natural notion to use in problems of reconstructing discontinuities. Recall that the concept of the wavefront set is an extension of the notion of the singular support of the function. The notion of the wavefront set goes beyond the definition of smoothness against non-smoothness of a function, i.e. the wavefront set not only provides the location of the non-smoothness of the function, but also the infinitesimal direction of that non-smoothness. The wavefront set is, of course, invariant under the addition of a smooth function.

In § 2 we state our main result (proposition 2.1) as a theorem for an integral operator. Its corollary (proposition 2.2) provides the wavefront set version of the answers to the questions (i) and (ii) stated above. A brief outline of how one arrives at such an integral operator in considering a linearised inverse scattering problem is given in § 3. In § 4 proposition 2.1 is proved by using some standard facts of the theory of pseudodifferential operators. These basic facts are summarised in the appendix.

2. Main result

Let X be an open subset of R^n with a smooth boundary ∂X. Let $\eta \in \partial X$ be fixed once and for all. Let $\varphi(x, \xi)$, $x \in X$, $\xi \in \partial X$ be a positive function in $C^\infty(X \times \partial X)$ satisfying the following conditions.

For each fixed $x \in X$, the map

$$(k, \xi) \to k \nabla_x \varphi(x, \xi) \tag{2.1}$$

from $R^+ \times \partial X$ onto a subset of $R^n \backslash 0$ is a diffeomorphism. Here, ∇_x is the gradient with respect to x and R^+ is the set of all positive real numbers.

For each fixed $\xi \in \partial X$, the map

$$x \to (\varphi(x, \xi) + \varphi(x, \eta), \nabla_\xi \varphi(x, \xi)) \tag{2.2}$$

from X onto a subset of R^n is a diffeomorphism. (∇_ξ is the gradient with respect to any local coordinates on ∂X.)

In addition, assume that two functions $A(x, \xi)$ and $B(x, \xi)$ satisfy the following conditions:

$$A(x, \xi) \in C^\infty(X \times \partial X) \qquad \text{and} \qquad A(x, \xi) > 0. \tag{2.3}$$

$$B(x, \xi) \in C^\infty(X \times \partial X) \tag{2.4}$$

and for each $x \in X$, B is compactly supported in ξ. We also assume that

$$\nabla_x(\varphi(x, \xi) + \varphi(x, \eta)) \neq 0 \tag{2.5}$$

for any (x, ξ) in the support of the function B in (2.4). Under these assumptions, we state the main result.

Proposition 2.1. Define an integral operator T: $E'(X) \to D'(X)$ (where D' is the space of distributions and E' is the space of distributions with compact support) by

$$g(x) = (Tf)(x) \tag{2.6}$$

where

$$(Tf)(x) = \int_{k=0}^{\infty} \int_{\partial X} \int_X f(y) \exp[\mathrm{i}k(\varphi(y, \xi) + \varphi(y, \eta) - \varphi(x, \xi) - \varphi(x, \eta))]$$
$$\times A(y, \xi) B(x, \xi)(-\mathrm{i}k)^{n-1} \, \mathrm{d}k \, \mathrm{d}\xi \, \mathrm{d}y.$$

Then

T is a pseudodifferential operator of degree 0 (modulo a smoothing operator)

T is elliptic at $(x_0, p_0) \in T^*X$ if and only if there exists $\xi_0 \in \partial X$ such that

$$B(x_0, \xi_0) \neq 0 \tag{2.7}$$

where

$$p_0 = -\nabla_x(\varphi(x_0, \xi_0) + \varphi(x_0, \eta)). \tag{2.8}$$

Here (x, p) are the local coordinates for the cotangent bundle T^*X of W with p being the dual coordinates to x. This proposition will be proved in § 4. As its corollary, we have

Proposition 2.2. For $g = Tf$ in proposition 2.1

$$WF(g) \subset WF(f) \tag{2.9}$$

$$\text{Point } (x_0, p_0) \in WF(f) \text{ belongs to } WF(g) \text{ if (2.7) is satisfied.} \tag{2.10}$$

This follows immediately from corollary A.2 in the appendix. The motivation for proposition 2.1 is given in the next section.

3. Linearised inverse scattering problems

Consider R^n as a physical medium where wave propagation is governed by a linear wave equation. Let $X \subset R^n$ be an open set with a smooth boundary ∂X. We denote by $n(x)$ the true index of refraction in X. (Recall that the index of refraction is the reciprocal of the wave velocity). Let $n_0(x)$ be an *a priori* known smooth positive function on X, regarded as a known part of the index of refraction, the so called background model. We assume that the medium is a small perturbation of the background model. Namely, we have on X,

$$n^2(x) = n_0^2(x) + f(x) \tag{3.1}$$

where we assume that f is bounded real and compactly supported in X and small so that the single scattering (or Born) approximation is valid. Our ultimate objective is to find the wavefront set of $n(x)$, or equivalently of $f(x)$. We assume that the ray structure on X generated by the background model is well behaved so that there exists a unique positive travel-time function (or phase)

$$\varphi(x, \xi) \in C^\infty(X \times \partial X)$$

satisfying the eikonal equation

$$|\nabla_x \varphi(x, \xi)|^2 = n_0^2(x) \tag{3.2}$$

for $x \in X$, and $\xi \in \partial X$, such that

$$\varphi(x, \xi) \to 0 \qquad \text{as } x \to \xi. \tag{3.3}$$

We also assume that $\varphi(x, \xi)$ satisfies conditions (2.1), (2.2). For example if $n_0(x)$ is constant, say $n_0(x) = 1$—the constant background model—we have

$$\varphi(x, \xi) = |x - \xi|$$

and all the necessary conditions are satisfied if X is convex.

For variable background model $\varphi(x, \xi)$ is the travel time from x to ξ along a ray in this model. We transmit a pure impulse $\delta(t)$, the delta function of time t, from the fixed source location $\eta \in \partial X$ and measure the scattered field for all time along the surface ∂X. (Recall that the total field is the sum of the incident field due to the background model and the scattered field. If one measures the total field at a point, then one can compute, at least in principle, the scattered field there because the incident field is known.) Using a geometrical optics approximation and the linearisation of the scattering problem (a first-order Born approximation), the Fourier transform with respect to time of the scattered field at $\xi \in \partial X$ and the real wavenumber k is given by (cf [10] (2.12), (3.1))

$$p(\xi, k) = (-\mathrm{i}k)^{n-1} \int_X f(y) \exp(\mathrm{i}k(\varphi(y, \xi) + \varphi(y, \eta)))A(y, \xi)\, \mathrm{d}y \tag{3.4}$$

where A is the product of two amplitude functions which are solutions to the transport equations, but here it suffices to note only that A satisfies (2.3).

Example. In the constant background case ($n_0(x) = 1$), we have

$$\varphi(x, \xi) = |x - \xi|$$

and

$$A(x, \xi) = \frac{1}{|x - \xi||x - \eta|}$$

(up to a positive constant). Thus (3.4) becomes the Lippman–Schwinger equation

$$p(\xi, k) = (-\mathrm{i}k)^{n-1} \int_X f(y) \exp(\mathrm{i}k(|y - \xi| + |y - \eta|)) \frac{1}{|y - \xi||y - \eta|}\, \mathrm{d}y.$$

We would like to recover the wavefront set of $f(x)$ from the observable $p(\xi, k)$, $\xi \in \partial X$, $k \in R$. Such partial inversion formulae are called migration algorithms and take the form

$$g(x) = \int_{k=0}^{\infty} \int_{\partial X} p(\xi, k) B(x, \xi) \exp(-\mathrm{i}k(\varphi(x, \xi) + \varphi(x, \eta)))\, \mathrm{d}k\, \mathrm{d}\xi \tag{3.5}$$

where B is any function satisfying (2.4) and (2.5). (For example, if X is convex and $\varphi(x, \xi) = |x - \xi|$, then the condition (2.5) is satisfied as long as $B(x, \xi) = 0$ for the antipodal point ξ to η with respect to x.) In some sense, g in (3.5) is regarded as a partial reconstruction of the desired function f. One wants to know the extent to which g approximates the true f. In particular one would like to know the relationship between the wavefront set of f and g. Substituting (3.4) into (3.5) one arrives at the equation $g = Tf$, where T is as in (2.6). This is the main motivation for studying the operator T in proposition 2.1. Thus from the viewpoint of inverse scattering problems, proposition 2.2

states that the locations of discontinuities together with their infinitesimal directions can be (at least partially) recovered. In fact the migration schemes described in [13–21] all have various choices for B, motivated in simple cases (e.g. a constant background) by some kind of explicit solution and in more complicated situations by physical or intuitive considerations. In [10–13] the choices of the function B are made so as to make the difference $g-f$ smoother than the function f itself. This allows one to recover the size of the jump in the case of a jump discontinuity. In this sense the result of [10] is slightly stronger than proposition 2.1.

Proposition 2.1 also explains why intuitive considerations have worked in applications, for example seismic imaging, and survived quite well without any underlying mathematical theory. All physical or intuitive considerations [16–22] were based upon the correct choice of travel-time function which as we can now see suffices to find the correct wavefront sets of the index of refraction and in turn is sufficient for practical needs in many cases.

4. Proof

We will prove our main result proposition 2.1. Using the change of variables

$$(k,\xi)\to\theta=k(\nabla_x(\varphi(x,\xi)+\varphi(x,\eta)))$$

which is valid globally on the support of B because of (2.1) and (2.5), we can write (2.6) as

$$g(x)=\int_{R^n}\int_X f(y)\exp(\mathrm{i}\Phi(x,y,\theta))C(x,y,\theta)\,\mathrm{d}\theta\,\mathrm{d}y. \tag{4.1}$$

Here the following notations are used:

$$C(x,y,\theta)=A(y,\xi)B(x,\xi)(-\mathrm{i})^{n-1}\left|\det\begin{pmatrix}\nabla_x(\varphi(x,\xi)+\varphi(x,\eta))\\ \nabla_\xi\nabla_x\varphi(x,\xi)\end{pmatrix}\right|^{-1} \tag{4.2}$$

where the determinant is non-zero due to (2.2), and

$$\Phi(x,y,\theta)=k(\varphi(y,\xi)+\varphi(y,\eta)-\varphi(x,\xi)-\varphi(x,\eta)). \tag{4.3}$$

Since $C(x,y,\theta)$ does not have the radial variable k in its expression, one easily checks from (2.3) and (2.4) that

$$C\in S^0(X\times X\times R^n\backslash\{0\}).$$

(See the remark after definition in appendix.) Also Φ satisfies all the conditions (A.1)–(A.6) of appendix for this C. To check it we note that (A.1) follows from the definition (4.3); (A.2) follows from (2.2); (A.3) follows from (2.5); (A.4) is clear from the definition (4.3); (A.5) follows from (2.2) and (A.6) follows from (2.1). Thus we may apply proposition A.1 (formulated in appendix) to the operator T in (2.6). Then proposition 2.1 follows immediately.

Acknowledgment

The work of PMC was supported partially by the NSF grant EAR-85-05922 and US Navy grant DNNUSC CU 00417601. W Chang and P Carrion would like to thank Dr A Quazi and Dr von Winkle for their encouragement and support.

Appendix

Here we state the basic facts about pseudodifferential operators. Let X be an open set of R^n.

Definition (cf [7] vol. 1, p 13). The space of symbols of degree m, denoted by

$$S^m(X \times X \times R^n \backslash \{0\})$$

consists of all complex-valued functions

$$C(x, y, \theta) \in C^\infty(X \times X \times R^n \backslash \{0\})$$

such that for each compact set $K \subset X \times X$, and multi-indices α, β, γ, there exists a constant $M_{\alpha,\beta,\gamma}(K)$ such that

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma C(x, y, \theta)| \leqslant M_{\alpha,\beta,\gamma}(K)(1+|\theta|)^{m-|\gamma|}$$

for all $(x, y) \in K$ and $\theta \in R^n \backslash \{0\}$. In some applications, the inequality is uniformly satisfied only for $|\theta| \geqslant \delta$, where δ is a positive constant. The essential part of the theory does not undergo any changes in such a case.

For the remainder, let $C \in S^m(X \times X \times (R^n \backslash \{0\})$ be fixed and D be a suitably small open set of $X \times X \times (R^n \backslash \{0\})$ containing supp C. We assume that there exists a function Φ satisfying the following conditions (A.1)–(A.6),

$$\Phi(x, y, \theta) \in C^\infty(D), \tag{A.1}$$

Φ is real, positive homogeneous of degree one in θ that is,

$$\Phi(x, y, \lambda\theta) = \lambda\Phi(x, y, \theta) \qquad \lambda > 0$$

$$\text{Set } \Lambda = \{(x, y, \theta) \in D, \nabla_\theta \Phi = 0\}. \tag{A.2}$$

Then $\Lambda = \{(x, x, \theta) \in D\}$

$$\nabla_x \Phi \neq 0 \qquad \nabla_y \Phi \neq 0 \qquad \text{on } \Lambda \tag{A.3}$$

$$\nabla_x \Phi = -\nabla_y \Phi \qquad \text{on } \Lambda \tag{A.4}$$

$$\det\left(\frac{\partial^2 \Phi}{\partial x \partial \theta}\right) \neq 0 \qquad \text{on } \Lambda. \tag{A.5}$$

If $\nabla_x \Phi(x, x, \theta) = \nabla_x \Phi(x, x, \theta')$, $(x, x, \theta) \in D$, $(x, x, \theta') \in D$, then $\theta = \theta'$. (A.6)

From this we get the following proposition.

Proposition A.1 (cf [7] vol. 2, p 463 remark 6.1). The Fourier integral operator $T: E'(X) \to D'(X)$ defined by

$$(Tf)(x) = \int \exp(\mathrm{i}\Phi(x, y, \theta)) C(x, y, \theta) f(y)\, \mathrm{d}y\, \mathrm{d}\theta \qquad f \in E'(X)$$

is in fact a pseudodifferential operator (modulo a smoothing operator). Its principal symbol at $(x, p) \in X \times R^n \backslash \{0\} = T^*(X)$ is given (up to a positive constant) by

$$C(x, x, \theta) \left| \det \frac{\partial^2 \Phi}{\partial x \partial \theta}(x, x, \theta) \right|^{-1}$$

where

$$p = \nabla_x \Phi(x, x, \theta).$$

Since the determinant factor is non-zero and positive homogeneous degree 0 in p, we conclude that T is elliptic at (x_0, p_0) if

$$|C(x, x, \theta)| \geqslant d|\theta|^m$$

for all (x, θ) in some conic neighbourhood of (x_0, θ_0) where

$$p_0 = \nabla_x \Phi(x_0, x_0, \theta_0)$$

and d is some positive constant. From this we get the following corollary.

Corollary A.2. Let $Tf = g$ be as in proposition A.1, then

(i) $WF(g) \subset WF(f)$

(ii) Let

$$(x_0, p_0) \in WF(f) \qquad p_0 = \nabla_x \Phi(x_0, x_0, \theta_0)$$

$$\text{if } |C(x, x, \theta)| > d|\theta|^m \qquad \text{for all } (x, \theta)$$

in some conic neighbourhood of (x_0, θ_0), where d is a positive constant, then we have

$$(x_0, p_0) \in WF(g).$$

Statement (i) follows from the pseudo-local property of T ([7], vol. 1, theorem 2.2) and (ii) follows from the micro-local regularity for an elliptic operator ([7], vol 2, proposition 6.10).

We note that for our main results (propositions 2.1, 2.2), we use the case $m = 0$ for proposition A.1 and corollary A.2.

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