1. Problem 1

(a) We have
\[ a_n - a_{n-1} - a_{n-2} = 0, a_0 = a_1 = 1. \]
The characteristic equation is
\[ r^2 - r - 1 = 0 \]
with roots \( r_1 = \frac{1-\sqrt{5}}{2}, r_2 = \frac{1+\sqrt{5}}{2} \). Thus,
\[ a_n = c_1 \left[ \frac{1-\sqrt{5}}{2} \right]^n + c_2 \left[ \frac{1+\sqrt{5}}{2} \right]^n. \]
Using the “initial conditions” we compute
\[ c_1 = -\frac{1-\sqrt{5}}{2\sqrt{5}} \text{ and } c_2 = \frac{1+\sqrt{5}}{2\sqrt{5}}. \]
Finally, compute the desired formula
\[ a_n = \frac{1}{2^{n+1}\sqrt{5}} \left[ (1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1} \right]. \]

(b) We multiply the given equation by \( n! \) and we get
\[ (n + 2)!a_{n+2} - 2(n + 1)!a_{n+1} - 3n!a_n = 0. \]
Then, we change the unknown using the notation \( b_n = n!a_n \) and we get the following linear homogeneous recursion relation with constant coefficients
\[ b_{n+2} - 2b_{n+1} - 3b_n = 0, b_0 = b_1 = 2. \]
We solve it easily using the characteristic equation
\[ r^2 - 2r - 3 = 0 \]
with roots \( r_1 = -1 \) and \( r_2 = 3 \). Then, \( b_n = c_1(-1)^n + c_23^n \). Using the “initial conditions” we get \( c_1 = c_2 = 1 \). Hence, \( b_n = (-1)^n + 3^n \).
Finally, we get the desired (since \( a_n = b_n/n! \))
\[ a_n = \frac{(-1)^n + 3^n}{n!}. \]

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The reason the provided code does not generate the desired B-spline is that Matlab's routine `spline` in this case defaults to generating not-a-knot spline. If we replace the line `yi = spline(x,y,x1);` by the line `yi = spline(x,[0,y,0],x1);`, we get the desired B-spline. This, of course, needs argumentation. First, we note that the B-spline is a natural spline which also has zero slopes at the ends of the interval (in our case the interval is \([-2,6]\)). However, the new line of code generates the complete spline on the same interval, with the same nodal values and with zero slopes at the ends of the interval. We just have to explain why the B-spline as a natural spline coincides with the described complete spline on \([-2,6]\) (we care only about \([-2,6]\) and we claim nothing about these splines and whether they coincide outside this interval).

Let \(S\) be the set of all twice continuously differentiable functions on \([-2,6]\) that take the desired values at the nodes and have zero slopes at the ends of the interval. Then, our complete spline (as a complete spline) is the unique function in \(S\) whose second derivative minimizes the \(L_2[-2,6]\) norm of all second derivatives of functions in \(S\). Consider the set \(B\) of all twice continuously differentiable functions on \([-2,6]\) that take the desired values at the nodes. In general, \(S \subseteq B\). Here, our B-spline (as a natural spline) is the unique function in \(B\) whose second derivative minimizes the \(L_2[-2,6]\) norm of all second derivatives of functions in \(B\). Using the obvious fact that the B-spline (having zero slopes at the ends of the interval) is in \(S\) and the uniqueness of the minimizers, we are left with nothing but to conclude that the described complete spline and the B-spline coincide (they are the same spline) on \([-2,6]\).

The not-a-knot and the respective B-spline are shown in Fig. 2.0.1.

3. **Problem 3**

Here we use the following representation of cubic splines with nodes \(x_0, x_1, \ldots, x_n\)

\[
s(x) = p_3(x) + \sum_{i=1}^{n-1} \beta_i (x - x_i)^3_+.
\]
Figure 3.0.2. Fastest descending cubic spline from 1 to 0 on an unit-spaced grid.

For our problem we use \( x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3, x_5 = 4 \). The node choice simplifies the exposition, but the result is general and does not depend on the location of our cubic spline on the \( x \)-axis when unit-spaced nodes are considered. Note that we could apply the constructive approach that we used in class for B-splines and actually construct the fastest descending from 1 to 0 spline by utilizing the technique using overdetermined linear systems. However, now we are readily given that the spline descends in 3 intervals so we look straight to the interval \([3,4]\). We have that \( s(x) \equiv 0 \) on \([3,4]\), where

\[
s(x) = 1 + \beta_1 x^3 + \beta_2 (x-1)^3 + \beta_3 (x-2)^3 + \beta_4 (x-3)^3, \text{ for } x \in [3,4].
\]

Taking the coefficients respectively in front of \( x^3, x^2, x^1, x^0 \) and setting them to zero, we get the following linear system for \( \beta_1, \ldots, \beta_4 \).

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & -3 & -6 & -9 \\
0 & 3 & 12 & 27 \\
0 & -1 & -8 & -27
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
-1
\end{bmatrix}.
\]

Note that, unlike the B-spline case, the system is non-homogeneous but also, unlike the B-spline case, it is non-singular and instead of looking at the null space we can simply solve it and get \( \beta_1 = -\frac{1}{6}, \beta_2 = \frac{1}{2}, \beta_3 = -\frac{1}{6}, \beta_4 = \frac{1}{6} \). Having these coefficients, we practically have the desired spline \( s(x) \) so we can plot it as seen in Fig. 3.0.2. Also, we compute the desired values \( s(1) = \frac{5}{6} \) and \( s(2) = \frac{1}{6} \).

4. Problem 4

Consider the cubic B-spline \( f(x) \) on a unit-spaced grid with support \([-2,2]\). That is, we have (as given in the figure in the assignment) \( f(-2) = f(2) = 0, f(-1) = f(1) = 1/6, \) and \( f(0) = 2/3 \). Also, \( f(x) \) can be viewed as a natural spline on \([-2,2]\), i.e. \( f\textquoteright\textquoteright\textquoteright(-2) = f\textquoteright\textquoteright\textquoteright(2) = 0 \). Define

\[
g(x) = \frac{1}{8} \left[ f(2x - 2) + 4f(2x - 1) + 6f(2x) + 4f(2x + 1) + f(2x + 2) \right].
\]

We want to show that \( f(x) \equiv g(x) \) everywhere. In general, natural splines on an interval can be uniquely \( C^2 \)-smoothly extended outside the interval by straight lines (without curvature, so with zero second derivatives). That is, we now only need to show that \( g(x) \) coincides with \( f(x) \) on \([-2,2]\) (i.e. that they represent the same unique natural spline) and automatically we will get that \( f(x) \equiv g(x) \) everywhere.
First, note that since \( f \in C^2(\mathbb{R}) \), clearly also \( g \in C^2(\mathbb{R}) \). Since \( f(x) \) is even (it is symmetric with respect to the \( y \)-axis), we have

\[
g(-x) = \frac{1}{8} [f(-2x - 2) + 4f(-2x - 1) + 6f(-2x) + 4f(-2x + 1) + f(-2x + 2)]
\]

\[
= \frac{1}{8} [f(2x + 2) + 4f(2x + 1) + 6f(2x) + 4f(2x - 1) + f(2x - 2)] = g(x),
\]

i.e. \( g(x) \) is also even (symmetric with respect to the \( y \)-axis). That is, we can concentrate on the interval \([0, 2]\).

Next, we compute

\[
g(0) = \frac{1}{8} [f(-2) + 4f(-1) + 6f(0) + 4f(1) + f(2)] = \frac{1}{8} \left[ 8 \cdot \frac{1}{6} + 6 \cdot \frac{2}{3} \right] = \frac{2}{3} = f(0),
\]

\[
g(1) = \frac{1}{8} [f(0) + 4f(1) + 6f(2) + 4f(3) + f(4)] = \frac{1}{8} \left[ \frac{2}{3} + 4 \cdot \frac{1}{6} \right] = \frac{1}{6} = f(1),
\]

\[
g(2) = \frac{1}{8} [f(2) + 4f(3) + 6f(4) + 4f(5) + f(6)] = 0 = f(2),
\]

\[
g''(x) = \frac{1}{8} [4f''(2x - 2) + 16f''(2x - 1) + 24f''(2x) + 16f''(2x + 1) + 4f''(2x + 2)]
\]

\[
= \frac{1}{2} \left[ f''(2x - 2) + 4f''(2x - 1) + 6f''(2x) + 4f''(2x + 1) + f''(2x + 2) \right],
\]

and

\[
g''(2) = \frac{1}{2} \left[ f''(2) + 4f''(3) + 6f''(4) + 4f''(5) + f''(6) \right] = 0 = f''(2).
\]

The symmetry allows us to conclude that all above holds for the points in \([-2, 0]\).

Clearly, \( g(x) \) is piecewise cubic. Actually, since we have coefficient 2 in front of \( x \) in the arguments of \( f(\cdot) \) in (1), we see that \( g(x) \) is clearly a cubic spline but on a equispaced grid with half-unit interval length. To show that \( g(x) \) is also a cubic spline on the unit-long intervals, we will prove that the third derivative is continuous (constant) on these intervals.

First, we have the unit-spaced B-spline computed in closed form during lectures (similarly to Problem 1 here) and using that we get the following

\[
f'''(x) = 6 \text{ on } (-2, -1),
\]

\[
f'''(x) = -18 \text{ on } (-1, 0),
\]

\[
f'''(x) = 18 \text{ on } (0, 1),
\]

\[
f'''(x) = -6 \text{ on } (1, 2)
\]

Namely, the closed form of the B-spline on \([-2, -1]\) is \((x + 2)^3\), on \([-1, 0]\) is \((x + 2)^3 - 4(x + 1)^3\). We can compute the third derivative on these intervals and get its value for the respective symmetric intervals by alternating the sign.

Now, for \(-2 < x < -3/2\) we have

\[
g(x) = \frac{1}{8} f(2x + 2),
\]

and

\[
g'''(x) = f'''(2x + 2) = 6.
\]

Similarly, for \(-3/2 < x < -1\) we have

\[
g(x) = \frac{1}{8} [f(2x + 2) + 4f(2x + 1)],
\]

and

\[
g'''(x) = f'''(2x + 2) + 4f'''(2x + 1) = -18 + 4 \times 6 = 6.
\]
Thus, \(g(x)\) is a cubic polynomial on \([-2, -1]\).

Furthermore, for \(-1 < x < -1/2\) we have
\[
\frac{1}{8}[6f(2x) + 4f(2x + 1) + f(2x + 2)],
\]
and
\[
g''(x) = 6f''(2x) + 4f''(2x + 1) + f''(2x + 2) = 6 \times 6 - 4 \times 18 + 18 = -18.
\]
Similarly, for \(-1/2 < x < 0\) we have
\[
g(x) = \frac{1}{8}[4f(2x - 1) + 6f(2x) + 4f(2x + 1) + f(2x + 2)],
\]
and
\[
g(x) = 4f''(2x - 1) + 6f''(2x) + 4f''(2x + 1) + f''(2x + 2) = 4 \times 6 - 6 \times 18 + 4 \times 18 - 6 = -18.
\]
Thus, \(g(x)\) is a cubic polynomial on \([-1, 0]\).

By symmetry, \(g(x)\) is a cubic polynomial also on the intervals in \([0, 2]\) and hence a cubic spline on the unit-spaced grid.

Having everything above and using the uniqueness of the natural spline we conclude that \(g(x) \equiv f(x)\) on \([-2, 2]\) and thus on the whole \(\mathbb{R}\).