2) We are given the eqns
\[
\begin{align*}
ax + by + cz &= 3 \\ ax - by + cz &= 1 \\ x + by - cz &= 2.
\end{align*}
\]
(1)

Our task is to compute \(a, b, c\) s.t., when solved, the system yields \(x=1, y=2, z=-1\). When the desired solution is plugged into (1), we obtain the system
\[
\begin{align*}
3a + 2b - c &= 3 \\ a - c &= 3 \\ 2b + c &= 7.
\end{align*}
\]
(2)

(2) can furthermore be represented as
\[
\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix},
\]
(3)

When (3) is solved using Gaussian Elimination, this system gives
\[
\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}.
\]

Switching to pen...

3) In this problem, we must find \(x, y, z, w\) s.t.
\[
\begin{pmatrix} x+y & x-z \\ y+w & x+2w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},
\]
(4)

To do this, set each element equal to each other. This gives the following system:
Finally, through Gaussian Elimination, one arrives at:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 1 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
W \\
X \\
Y \\
Z \\
\end{pmatrix}
= 
\begin{pmatrix}
2/3 \\
-1/3 \\
4/3 \\
-1/3 \\
\end{pmatrix}
\]

12)

(a) Claim: If \( D = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) is a \( 2 \times 2 \) diagonal matrix with \( a \neq b \), then the only matrices that commute \( w/ D \) are other \( 2 \times 2 \) matrices.

\( \textbf{Pf} \) Let \( A = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \). Then \( DA = \begin{pmatrix} aw & ax \\ by & bz \end{pmatrix} \) and \( AD = \begin{pmatrix} aw & bx \\ ay & bz \end{pmatrix} \). We need \( DA = AD \), so we set these products equal to each other element-wise, giving the following set of equalities:

\[
\begin{align*}
aw &= aw \\
ax &= bx \\
by &= ay \\
bz &= bz
\end{align*}
\]

Here, we recall \( a \neq b \) which implies \( x = y = 0 \). Therefore, the only matrices that commute \( w/ D \) take the form of \( A = \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix} \), where it is apparent that \( A \) is indeed \( 2 \times 2 \) diagonal.

(b) Now we must consider the possibility that \( a = b \). From the equalities in (a), it is observed that all \( 2 \times 2 \) matrices commute \( w/ D \) in this case.

(c) We must now find all matrices which commute \( w/ D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). We are given the condition that \( a \neq b \neq c \). Here the solution method is exactly the same as in part (a). With that being said, let \( A = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \). Then the products \( DA \) and \( AD \) are given by:

\[
AD = \begin{pmatrix}
a x_{11} & b x_{12} & c x_{13} \\
a x_{21} & b x_{22} & c x_{23} \\
a x_{31} & b x_{32} & c x_{33}
\end{pmatrix}
\]

\[
DA = \begin{pmatrix}
a x_{11} & a x_{12} & a x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{pmatrix}
\]

We must set these equal to each other element-wise.
Doing this, in turn, gives the set of equalities:
\[
\begin{align*}
ax_{21} &= bx_{21}, \\
ax_{31} &= cx_{31} \\
bx_{12} &= ax_{12}, \\
bx_{32} &= cx_{32} \\
cx_{13} &= ax_{13}, \\
cx_{23} &= bx_{23},
\end{align*}
\]
Utilizing the fact that \(a \neq b \neq c\), the above equalities show us that all of the off-diagonal elements in \(A\) are zero, i.e.
\[
A = \begin{pmatrix}
x_{11} & 0 & 0 \\
0 & x_{22} & 0 \\
0 & 0 & x_{33}
\end{pmatrix}
\]
Thus, all diagonal 3x3 matrices commute with \(D\).

(d) Now, if \(a = b = c\), the aforementioned equalities give a slightly different matrix \(A\). This time, the non-diagonal elements of the first row and column are equal to zero, and \(x_{23}\) and \(x_{32}\) are free to be whatever their hearts desire:
\[
A = \begin{pmatrix}
x_{11} & 0 & 0 \\
0 & x_{22} & x_{23} \\
0 & x_{32} & x_{33}
\end{pmatrix}
\]

(a) Here we are held responsible for finding all solutions \(X = (x, y, w)^T\) to the matrix eqn. \(AX = XB\) for \(A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\) and \(B = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}\). Let us begin by taking the products \(AX\) and \(XB\):
\[
AX = \begin{pmatrix} x + 2z \\ -x - y \end{pmatrix}, \quad XB = \begin{pmatrix} 3y \\ 3w \end{pmatrix}
\]
Setting the elements of these matrices equal to one another, we are presented with the following system of eqns:
\[
\begin{align*}
x - 3y + 2z &= 0 \\
x + y + 2w &= 0 \\
x + 3w &= 0 \\
y + z &= 0.
\end{align*}
\]
We can (and most definitely should) put this into matrix form:
\[
\begin{pmatrix}
1 & -3 & 2 & 0 \\
-1 & 1 & 0 & 2 \\
1 & 0 & 0 & 3 \\
0 & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
The solution of this system is the zero vector, and therefore \[ X = (0, 0) \] 

(b) Here's an example of a pair of non-zero matrices \( A \neq B \) s.t. the previously solved matrix equ does not have a trivial soln:

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

The solution gives \( X = (1, 0) \), & it is apparent that \( AX = (0, 0) = XB \).

31) Commutators.

(a) If \( A \) is \( m_1 \times n_1 \), \( B \) is \( m_2 \times n_2 \), \( AB \) is only defined if \( n_1 = m_2 \) & \( BA \) is defined only if \( n_2 = m_1 \). Furthermore, \( AB \) is \( m_1 \times n_2 \) & \( BA \) is \( m_2 \times n_1 \), so \( AB - BA \) is only defined if \( m_1 = m_2 \) \& \( n_1 = n_2 \), so \( m_1 = m_2 = n_1 = n_2 \).

(b) Now we must show that \( A \) & \( B \) commute under matrix multiplication iff \( [A, B] = 0 \).

\[ [A, B] = AB - BA = AB - BA = 0 \]

\[ \iff [A, B] = 0. \]

If \( [A, B] = 0 \). Then

\[ AB - BA = 0 \]

\[ \Rightarrow AB = BA \]

\[ \therefore A \& B \text{ commute}. \]

(c) (i) \( \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \), (ii) \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), (iii) \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

(d) We must prove that the commutator of


So \[ [CA + dB, C] = (CA + dB)C - C(CA + dB) = CAC - CCA + dBC - dCB = C[A, C] + d[B, C]. \]

\[ [A, aB + dC] = A(aB + dC) - (aB + dC)A = aAB - aBA + dAC - dCA = C[A, B] + d[A, C]. \]


(2.11)

(\* iff = if \& only if, \( \iff \) = suppose, \therefore = therefore) (A \( \heartsuit \) denotes a desired equality/inequality)
(iii) satisfies the Jacobi identity. Need

\[ \star [A, [B, C]] + [B, [A, C]] + [C, [B, A]] = 0. \]

\[ \Rightarrow [[A, B], C] = [A B - B A, C] = [A B, C] - [B A, C] \quad \text{(by (i))} \]

\[ = A B C - C A B - B A C + C B A \]

\[ [[C, A], B] = [C A - A C, B] = [C A, B] - [A C, B] \]

\[ = C A B - B A C - A C B + B A C \]

\[ [[B, C], A] = [B C - C B, A] = [B C, A] - [C B, A] \]

\[ = B C A - A B C - C B A + A B C \]

\[ \Rightarrow \mathbf{A} = A B C - C A B - B A C + C B A + B C A - A C B + A B C - C B A + A C B \]

\[ = 0 \]

32)

(a) \( A_1 = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \), \( \text{Tr} \ A_1 = 1 + 3 = 4 \)

(b) Here we prove \( \text{tr} (A + B) = \text{tr} A + \text{tr} B \):

\[ \text{Pf: } \text{det} \ A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \]

Then \( \text{Tr} \ A = \sum_{i=1}^{n} a_{ii} \) and \( \text{Tr} \ B = \sum_{i=1}^{n} b_{ii} \).

\[ A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nn} + b_{nn} \end{pmatrix} \]

\[ \text{Tr} (A + B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) \]

\[ = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \]

\[ (c) \text{ And mean to investigate } \text{tr} (A B) = \text{tr} (B A). \]

\[ A_{i j} = a_{i j}, \quad B_{i j} = b_{i j}. \text{ Where } i \text{ is a row, } j \text{ is a column.} \]

Then \( (A B)_{i j} = \sum_{k=1}^{n} a_{i k} b_{k j} \).

\( \text{We know } \text{tr} (A B) = \sum_{i=1}^{n} (A B)_{i i} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{i k} b_{k i}, \)
and similarly, \( \text{tr}(BA) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ki} \).

If we take the double sum that is \( \text{tr}(BA) \) and reindex it, switching \( i \) & \( k \), we get

\[
\text{tr}(BA) = \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ki} b_{ki} = \text{tr}(AB).
\]

(d) Claim: If \( C = AB - BA \), then the matrix \( C \) has \( \text{tr}(C) = 0 \).

\[\text{Pf} \quad \text{tr}(C) = \text{tr}(AB) + \text{tr}(BA) \quad \text{by (b)} \]
\[= \text{tr}(AB) + \text{tr}(-AB) \quad \text{by (c)} \]
\[= \text{tr}(AB - AB) \]
\[= \text{tr}(0) \quad \text{by (c)} \]
\[= 0 \text{ (by definition)} \]

Note that this problem was about commutators.

(e) Turns out that (c) is indeed valid if \( A \) is \( m \times n \) & \( B \) is \( n \times m \):

\[
\text{tr}(AB) = \sum_{i=1}^{m} (AB)_{ii}; \quad \text{tr}(BA) = \sum_{i=1}^{n} (BA)_{ii} = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ki} b_{ki} \]
\[= \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ki} b_{ki} \quad \text{by (c)} \]
\[= \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ki} b_{ki} = \text{tr}(AB). \]

33) \( \hat{x} = \sum_{i=1}^{n} c_i \hat{c}_i = \hat{y} \)

The \( k^{th} \) entry of \( \hat{y} \) is \( x_k a_{k1} + x_k a_{k2}^2 + \cdots + x_k a_{kn} \).

Which is the usual method of matrix multiplication. There you have it, justification for this alternative formula.

34)

(a) \( M = \begin{bmatrix} i[A] & i[B] \\ i[C] & i[D] \end{bmatrix} \Rightarrow M \text{ is } (i+j) \times (k+l) \)

(b) \( M = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \)
Claim: If \( N = (P Q) \) is a block matrix whose blocks have the same size as those of \( M \), then
\[
M + N = \begin{pmatrix}
A + P & B + Q \\
C + R & D + S
\end{pmatrix}.
\]

**Proof:***
\[
M = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1k} \\
& \ddots & & \vdots \\
& & \ddots & \vdots \\
& & & a_{kk}
\end{pmatrix}
\]
\[
N = \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1k} \\
& \ddots & & \vdots \\
& & \ddots & \vdots \\
& & & b_{kk}
\end{pmatrix}
\]

Then
\[
M + N = \begin{pmatrix}
a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1k} + b_{1k} \\
a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2k} + b_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{k1} + b_{k1} & a_{k2} + b_{k2} & \cdots & a_{kk} + b_{kk}
\end{pmatrix}
\]

(d) \( A \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} Ax \\ Ay \end{pmatrix} \)

To form the \( ij^{th} \) entry: if row \( i \) lies across \( A \) & \( B \), then it is the concatenation of two vectors \( \hat{a}_i, \hat{b}_i \). Similarly, if column \( j \) lies across \( X \) & \( Z \), it is the concatenation of two column vectors \( \hat{x}_j, \hat{z}_j \).

The vectors are multiplied to get the scalar \((MP)_{ij} \): 
\[
\begin{bmatrix}
a_{i1} & \cdots & a_{ik} & b_{i1} & \cdots & b_{ik}
\end{bmatrix}
\begin{bmatrix}
x_{j1} \\
x_{j2} \\
\vdots \\
x_{jk}
\end{bmatrix}
= (AX)_{ij} + BZ_{ij}
\]

Repeat this procedure on the other 3 blocks.
So two block matrices are compatible if all of the products $AX, BY, AY, BW, CX, DZ, CY, DW$ are defined, as are the sums $A + B, A + B, C + D, C + D, C + D$.

Let $X = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $Y = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $Z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $W = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. (Many answers here, provided is an example of matrices which would work.)

Validation is left to the reader.

§ 1.3

13. We must show that $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent.

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) Claim: any strictly upper triangular matrix is nilpotent.

Proof. Induction.

Take for the base case the matrix $[0]$. This is the only strictly upper triangular matrix, so it is trivially nilpotent.

Inductive step: If all $n \times n$ strictly upper triangular matrices are nilpotent, let $A$ be an arbitrary matrix of this type, so $A^k = 0$.

We form the block matrix

$$M = \begin{pmatrix} A & M_{n \times m} \\ \cdots & \cdots \end{pmatrix},$$

with

$$M^k = \begin{pmatrix} A^k & \left[ \prod_{i=1}^{k-1} A \right] B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

by the inductive hypothesis.

\[ \therefore \text{ All } n \times n \text{ strictly upper triangular matrices } A \text{ are nilpotent.} \]

*(\prod \text{ denotes the continued product)*
22) Let's find the LU factorization of 
\[ A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \]

Here we go...

\[ R_2 = R_2 - \frac{1}{2} R_1 \]
\[ \Rightarrow E_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & -1/2 \\ 0 & 1 & 1 \end{pmatrix} \]

\[ R_3 = R_3 - \frac{1}{3} R_2 \]
\[ \Rightarrow E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix}, \quad A = E_2^{-1} A E_2 = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ L_1 = E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}, \quad L_2 = E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \]

Verification is left for the reader.

31) We must solve \( A \bar{x} = \vec{b} \) using \( L, U \) from 22(d) for \( \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

\[ A \bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow LU \bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

First, solve \( \bar{L} \bar{c} = \vec{b} \), to find that the coefficient vector \( \bar{c} \) is given by \( \bar{c} = \begin{pmatrix} 1/2 \\ 5/6 \end{pmatrix} \). Then solve \( U \bar{x} = \bar{c} \) to obtain \( \bar{x} = \begin{pmatrix} -4/7 \\ 2/7 \end{pmatrix} \). These vectors are solved for by the usual Gaussian Elimination Method.

#7 True/False

(a) The product of two diagonal matrices is diagonal.

Well, take \( A = (a_{ij}) \), \( B = (b_{ij}) \), \( a_{ij} = 0 \), \( i \neq j \)

\[ (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{ii} b_{jj} = 0 \] if \( i \neq j \), so \( AB \) is diagonal

\[ \boxed{\text{TRUE}} \]
(b) If $A$ and $B$ are two matrices with $AB = 0$, then either $A = 0$ or $B = 0$.

Um, take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

**False**

(c) If $A$ and $B$ are two matrices of the same size, then $AB = BA$.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

**False**

(d) If $A$ and $B$ are square matrices of the same size, then $A^2 - B^2 = (A+B)(A-B)$.

So, $(A+B)(A-B) = A^2 - AB + BA - B^2$. We observe that the quantity $BA - BA = 0$ if and only if $A$, $B$ commute.

**False**