1. One can readily verify that the cubic B-spline on unit-spaced nodes can be expressed in the form

\[
B(x) = \frac{1}{12}( |x+2|^3 - 4|x+1|^3 + 6|x|^3 - 4|x-1|^3 + |x-2|^3 )
\]

From this relation (1) follows that the Fourier transform of \( B(x) \) also takes a remarkably simple form:

\[
\hat{B}(\omega) = \frac{1}{2\pi} \left( \frac{2 \sin \frac{\omega}{2}}{\omega} \right)^4
\]

This can easily be derived by first noting that \( B(x) \) vanishes identically outside \([-2,2]\), i.e. the integration interval for the Fourier transform can be changed from \([-\infty, \infty]\) to \([-2,2]\). The integral \( \int_{-2}^{2} B(x) e^{i\omega x} dx \), using the representation (1), then becomes just the sum of five elementary integrals. This sum simplifies to (2).

a. Give a brief heuristic argument for why the asymptotic decay rate \( |\hat{B}(\omega)| = O\left(\frac{1}{\omega^4}\right) \), apparent from (2), is just what should be expected for when \( |\omega| \to \infty \).

b. The general properties of a Fourier transform given in Table 1, page 154, of the notes "Fourier Series and Transforms" on the class web page include that \( u(x) \leftrightarrow \hat{u}(\omega) \) implies \( u(x-a) \leftrightarrow e^{-i\omega a} \hat{u}(\omega) \). Find similarly what the Fourier transform is for \( u(2x) \) in terms of the one for \( u(x) \).

c. Show again (cf. Homework 4, Problem 4) that the cubic B-spline satisfies

\[
f(x) = \frac{1}{4}(f(2x-2) + 4f(2x-1) + 6f(2x) + 4f(2x+1) + f(2x+2))
\]

this time by using the results above to verify that the two sides of the equality have the same Fourier transform.

**Hint for c:** Mathematica can be helpful when doing the necessary simplifications.

2. Suppose we want to compute \( r = \sqrt{x^2 + y^2} \) when \( x \) and \( y \) are given. Consider the following geometric description of a possible iterative procedure (see the figure to the right):

i. Let \( x_0 = x, \ y_0 = y \).

ii. From the point \( A = (x_0, y_0) \), drop a line vertically down to the x-axis. Its midpoint is \( B \).

iii. Draw from the origin \( O \) the straight line \( OBC \).

iv. Find a point \( E = (x_1, y_1) \) such that \( AE \) is orthogonal to \( OC \) and the length of \( AD \) and \( DE \) are the same (clearly \( E \) is then located the same distance from the origin \( O \) as the original point \( A \)).

v. Repeat the same process to get \( (x_2, y_2) \) from \( (x_1, y_1) \), etc.

Your tasks are to:

a. Express the described iteration in standard algebraic (rather than geometric) form.

b. Demonstrate analytically and confirm with a Matlab test that the sequence of iterates \( \{x_k\}, \ k = 0, 1, 2, 3, \ldots \) converges **cubically fast** to \( r \).
3. Prove the first derivative recursion relations for orthogonal polynomials in case of
   a. Chebyshev polynomials: \( \frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1} = 2T_n \), and
   b. Legendre polynomials: \( P'_{n+1} - P'_{n-1} = (2n+1)P_n \).

Hints: For a, the easiest approach is probably to start off with one of the explicit formulas for \( T_n(x) \). For b, you might try to proceed in the style of how the three term recursions were demonstrated in class.

4. Taylor expansions often fail to converge on the real axis because of singularities that are located away from it in the complex plane. One way to arrive at Chebyshev expansions is to ask for what kind of polynomial bases the real-axis convergence will not require off-axis analyticity. It turns out that we thereby usually can increase the domain of convergence in the complex plane. If the interval of interest is [-1,1], a Chebyshev expansion will converge in the largest ellipse with foci at -1 and +1, and extending out to the first singularity. This result can be shown without much trouble from the formula

\[
T_n(x) = \frac{1}{2} (z^n + \frac{1}{z^n}) \tag{3}
\]

where \( z \) is implicitly defined through \( x = \frac{1}{2} (z + \frac{1}{z}) \). The elliptical domain guarantees that the full interval of interest on the real axis (here [-1,1]) is included, whereas a Taylor series can easily fail over part of the interval of interest.

a. Prove the relation given in equation (3).

b. In the case of \( \arctan x \), the Taylor and Chebyshev expansions become

\[
\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} \frac{2(-1)^k}{2k+1} (\sqrt{2} - 1)^{2k+1} T_{2k+1}(x) \tag{4}
\]

Computationally compare the convergence rates of these two expansions by truncating both of the sums at some choices for \( k = N \), and then plot their respective errors over [-1,1].