1 Question 1

\[ \nabla u = \frac{d}{dx} f(g) + \frac{d}{dy} f(g) = [f'(g) g_x, f'(g) g_y], \]
\[ \implies \nabla \times \nabla u = f'(g) g_x g_y - f'(g) g_y g_x = 0 \]

\[ \frac{\partial u}{\partial x} = \frac{\partial f}{\partial g} \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} \right), \]
\[ \frac{\partial u}{\partial y} = \frac{\partial f}{\partial g} \left( \frac{\partial g}{\partial y} + \frac{\partial g}{\partial u} \frac{\partial u}{\partial y} \right). \]

If \( 1 - f' g_u \neq 0 \), then
\[ u_x g_y = (1 - \frac{\partial f}{\partial g} \frac{\partial g}{\partial u})^{-1} \frac{\partial f}{\partial g} \frac{\partial g}{\partial x} \frac{\partial g}{\partial y} = (1 - \frac{\partial f}{\partial g} \frac{\partial g}{\partial u})^{-1} \frac{\partial f}{\partial g} \frac{\partial g}{\partial y} \frac{\partial g}{\partial x} = u_y g_x. \]

2 Question 2

\[ \frac{\partial \rho}{\partial t} + c(\rho) \frac{\partial \rho}{\partial x} = k \frac{\partial^2 \rho}{\partial x^2}, \]
\[ \implies \frac{\partial q}{\partial x} = \frac{\partial}{\partial x} \left( Q(\rho) - \kappa \frac{\partial \rho}{\partial x} \right) = Q'(\rho) - \kappa \frac{\partial^2 \rho}{\partial x^2} \]
\[ \text{where } Q'(\rho) = c(\rho). \]
3 Question 3

\[ u_{xx} = f''(x + ct) + g''(x - ct), \]
\[ u_{tt} = c^2(f''(x + ct) + g''(x - ct)), \]
\[ \Rightarrow u_{tt} = c^2 u_{xx} \]

\[ u_{tt} = c^2(G''(r - ct) + \frac{N - 1}{r}G'(r - ct)) \]
only if \( N = 1. \)

\[ \begin{align*}
F \text{ is constant} & \quad \text{if } N = 1, \\
F(r) = \frac{c}{r}, c \text{ is constant} & \quad \text{if } N = 3.
\end{align*} \]

\[ c^2FG'' = c^2[F''G + 2F'G' + FG'' + \frac{N - 1}{r}(F'G + FG')], \]
\[ \Rightarrow F'' + \frac{N - 1}{r}F' = 0, F' + \frac{N - 1}{2r}F = 0 \]

Solve \( F' + \frac{N - 1}{2r}F = 0, F = cr^{(1-N)/2} . \) Then plug into \( F'' + \frac{N - 1}{r}F' = 0, \) then \( N = 3, 1. \)

• Let \( F_1 = D \) and \( F_2 = H, \) then

\[ \Rightarrow 0 = D''G + 2D'G' + H''G' + 2H'G'' + \frac{N - 1}{r}(D'G + DG' + H'G' + HG''), \]
\[ \Rightarrow H = C_1 r^{(1-N)/2}, \]
\[ \Rightarrow D = C_2 r^{2-N} + C_3 \]
\[ \Rightarrow C_1 = \frac{1}{3}C_2C_3 = 0 \]
\[ \Rightarrow F_1(r) = -C_2 r^{-3}, F_2(r) = C_2 r^{-2}. \]

• If \( N = 1, \) \( F_1 \) and \( F_2 \) are constants. If \( N = 3, \) \( F_1(r) = -C_2 r^{-1} \) and \( F_2(r) = C_1 r^{-1}. \) We just solve for \( N = 5. \)

If \( N > 1 \) and \( N \neq 3, 5, \) then

\[ \frac{r^{2-N}}{r} = r^{1-N} \neq r^{-(N+3)/2} \Rightarrow N = 3 \]
4 Question 4

\[ \frac{G''}{G} + 1 = \frac{F''}{F} = k \]

Case 1, \( k > 0, k = \lambda^2, F = 0 \).
Case 2, \( k = 0, F = 0 \).
Case 3, \( k < 0, k = -\lambda^2, F = a_n \sin(\lambda_n x), \lambda_n = \frac{\mu}{E} \).

\[ G_n(t) = c_n \cos(\sqrt{1 + \lambda^2 t}) + D_n \sin(\sqrt{1 + \lambda^2 t}), \]
\[ U_n(x, t) = \sin(\lambda_n x)[c_n \cos(\sqrt{1 + \lambda^2 t}) + D_n \sin(\sqrt{1 + \lambda^2 t})], \]
\[ U(x, t) = \sum_{n=1}^{\infty} U_n(x, t) \]

5 Question 5

\[ \nabla \times (\nabla \times E) = \nabla \times (\mu \partial_t H) = -\mu \cdot \varepsilon \cdot \frac{\partial^2 E}{\partial t^2}, \]
\[ \Rightarrow \partial_{tt} E + c^2 \nabla \times (\nabla \times E) = 0. \]

\[ \nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E \]

If \( \nabla \cdot E = 0 \), then \( \partial_{tt} E = c^2 \nabla^2 E \)

6 Question 6

\[ u_2 = (\text{cost}) \partial_x u_1, \]
\[ u_3 = (\text{cost}) \partial_x u_2, \]

For 2-D case,

\[ u(x, y, t) = Ae^{-1/4 \left( \frac{(x-x_0)^2+(y-y_0)^2}{r^2} \right)} t^{-1}, \]

7 Question 7

\[ M(t) = \int_{-\infty}^{\infty} u(x, t)dx, \]
\[ \Rightarrow \partial_t M(t) = \int_{-\infty}^{\infty} \partial_t u(x, t)dx = \int_{-\infty}^{\infty} k \partial_x^2 u(x, t)dx, = 0. \]
\[ \partial_t M(t)X + M(t) \partial_t X(t) = \int_{-\infty}^{\infty} x \partial_t u(x, t) dx = \frac{k}{M} \int_{-\infty}^{\infty} x \partial_x^2 u(x, t) dx, \]

\[ = 0, \]

\[ \Rightarrow \partial_t X(t) = 0, \]

which means the center of mass is constant, and does not move as time varies.

\[ M(t) \partial_t L^2 = k \int_{-\infty}^{\infty} x^2 \partial_x^2 u(x, t) dx = 2M(t), \]

\[ \Rightarrow \partial_t L^2 = 2 \Rightarrow L = \sqrt{2t}, \]

which means the length of the spreading mass \( \rightarrow \infty \).

With this information, we should be able to determine the time when it starts to concentrate a certain area, depending on the location of the area versus the location of the insertion of the solute.
8 Guenther & Lee, Section 1.3 #1

\[ q \cdot n = -ku_x, \]
\[ -\text{div} q = ku_{xx}, \]
\[ u_t = au_{xx} + F, a = k/c\rho, F = f/c\rho. \]

9 Guenther & Lee, Section 1.3 #3

Same with (3−1) to (3−5),
(3−6) changes to \( c_t = c(x)u_t, \)
(3−7) changes to \( q = -k(x)\nabla u, \)
\[ \Rightarrow c(x)\rho u_t = \nabla \cdot (k(x)\nabla u) + f \]

10 Guenther & Lee, Section 1.7 #1

Start with
\[ \rho_t + \text{div}(\rho \nu) = 0, \]
\[ \rho [\nu_t + (\nu \cdot \nabla)\nu] = -\nabla p - \rho g, \]
\[ \frac{p}{\rho_0} = (\frac{\rho}{\rho_0})^\lambda = 1 + \lambda u \]
assume
\[ 1 + v = \frac{\rho}{\rho_0} \]
\[ \frac{p_t}{\rho_0\lambda} - \nabla \cdot \frac{\partial}{\partial t} v = 0, \]
\[ \Rightarrow p_{tt} = c^2 \Delta p, c^2 = \frac{p_0\lambda}{\rho_0}. \]

11 Guenther & Lee, Section 1.8 # 3

Let \( f_1(x) = 0 \) and \( g_1(x) = 0, \) and \( u_1(x,y) = 0. \) Then we can verify that \( u_1 \) and \( f_1, g_1 \) is the solution and initial condition to the PDE.

Then, let \( f_2(x) = \frac{1}{n} \cos(nx) \) and \( g_2(x) = 0, \) and \( u_2(x,y) = \frac{1}{n} \cos(nx) \cosh(ny). \)
Then, we have
\[ u_y(x,y) = \frac{1}{n} \cos(nx) \sinh(ny) \]
Similarly, we can verify that \( u_1 \) and \( f_1, g_1 \) is also the solution and initial condition to the PDE. Plus, we have \( \lim_{n \to \infty} |f_1(x) - f_2(x)| \) is uniformly about \( x. \)
Let $x = \pi$, we have

$$|u_1(\pi, y) - u_2(\pi, y)| = \left| \frac{\cosh(ny)}{n^2} \right| = \frac{e^{ny} + e^{-ny}}{2n^2}$$

then we have

$$\lim_{n \to \infty} |u_1(\pi, y) - u_2(\pi, y)| = \infty.$$  

Thus, when the initial condition becomes close, the solution will become more different. Therefore, the solution’s behavior largely changed when there is a slight change in the initial condition, which means that the PDE is an ill-posed one.