1. a. Starting from the nowhere (apart from at $z = 0$) convergent Taylor expansion of the Stieltjes function $f(z) = \int_0^\infty \frac{e^{-t}}{1 + z t} dt$, use the code `t2cf` below to generate from this expansion a sequence of leading continued fraction (CF) coefficients. Identify the CF coefficient pattern that emerges, and then use this pattern to evaluate $f(2)$ as accurately as you can.

Note: When starting from the integral formulation, everything is well conditioned, and Mathematica readily provides its value to any desired level of precision, for ex.

$$\text{N}\left[ \int_0^\infty \frac{e^{-t}}{1 + z t} dt \bigm/ z \to 2, 30 \right]$$

gives the result 0.461455316241865234416424687914

b. Same problem as above, but consider instead the function $f(z) = \frac{\log(1 + z)}{z}$.

Hint: Matlab's built-in function "rat" can be very helpful for converting floating point numbers to rational form.

c. Consider the function $f(z)$, given by the beginning of its Taylor expansion, as follows:

$$f(z) = 1 + z - z^2 + \frac{4}{3} z^3 - \frac{5}{4} z^4 + \frac{12}{15} z^5 - \frac{122}{45} z^6 + \frac{1088}{315} z^7 - \frac{227}{63} z^8 + \frac{15872}{2835} z^9 - \frac{101042}{14175} z^{10} + O(z^{11}).$$

Convert this expansion to its CF form, and attempt to spot a closed form expansion for its CF coefficients. Produce a plot that compares, over $z \in [-3, 3]$, (i) direct evaluation of the truncated Taylor series, (ii) a high order CF version, and (iii) the function $f(z) = 1 + \frac{1}{1 + \cot(z)}$ (which the given Taylor expansion in fact was obtained from).

Matlab codes for problem 1:

```matlab
function cf = t2cf(c)
    % Converts a truncated Taylor series (given as row vector in c) into a
    % continued fraction expansion of same size. This algorithm can fail
    % in some cases, notably if the Taylor expansion has any zero coefficients.
    n = size(c,2); a(:,1) = c.'; b = [1 zeros(1,n-1)]';
    for k=2:n
        a(1:n-k+1,k) = b(2:n-k+2)-a(2:n-k+2,k-1)/a(1,k-1);
        b(1:n-k+1)   = a(1:n-k+1,k-1)/a(1,k-1);
    end
    cf = a(1,:);
end

function y = cf2y(x,cf)
    % Evaluates a continued fraction expansion (in row vector cf) for the x-values
    % given in x (which may be a scalar, vector or array of any size/shape).
    y = zeros(size(x));
    x = x(:);
    len = size(cf,2);
    r = zeros (length(x),len+1); % Loop through the cont. fraction expansion.
    s = zeros (length(x),len+1);
    r(1,:) = cf(1); s(1,:) = 1;
    for k=2:len+1
        r(:,k) = r(:,k-1)+cf(k-1)*x.*r(:,k-2);
        s(:,k) = s(:,k-1)+cf(k-1)*x.*s(:,k-2);
    end
    y(:) = r(:,len+1)./s(:,len+1); % Result gets here re-arranged from column vector
end
```
2. A Gaussian quadrature formula for weight function $w(x) \geq 0$ takes the form

$$\int_a^b f(x) w(x) \, dx \approx \sum_{i=1}^n w_i f(x_i)$$

and is exact for all polynomials $f(x)$ of degree $2n-1$ or less. Show that all the weights $w_i$ are non-negative.

**Hint:** By considering some suitable test functions $f(x)$, the result follows very quickly.

3. Atkinson's Example 3 on pages 262-263 concerns $\int_0^{2\pi} e^{\cos(x)} \, dx \approx 7.95492652101284$. One way to understand the extremely high rate of convergence of the trapezoidal rule (and - to a lesser extent - Simpson's rule) for a periodic function such as this one starts by noting that the integrand can be Fourier expanded. In the present case

$$e^{\cos(x)} = \sum_{n=0}^{\infty} a_n \cos(nx) \quad \text{where} \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} e^{\cos(x)} \, dx \quad \text{and} \quad a_n = \frac{1}{\pi} \int_0^{2\pi} e^{\cos(x)} \cos(nx) \, dx, \quad n > 0.$$

By asymptotic analysis (the topic of APPM 5480), one can readily show that $\lim_{n \to \infty} a_n 2^{n-1} n! = 1$. Assuming (quite correctly) that this limit result also provides good approximations at low values of $n$, use this to derive approximations for the trapezoidal and Simpson errors when the original integral is discretized at $x_i = 2\pi i/8, \quad i = 0, 1, \ldots, 8$. Compare what you obtain against the values for this case that are quoted in Atkinson Table 5.7. Also tell how many nodes $n$ you would need for trapezoidal rule to give the very high accuracy of $10^{-60}$.

**Hint:** Simpson's rule can be seen as the Richardson extrapolation of two trapezoidal evaluations based on steps $h$ and $h/2$, respectively, i.e. linearly combining the results so that the leading $O(h^2)$ error term vanishes.

4. Gaussian quadrature is mostly used for accurate evaluation of integrals. One useful generalization is to instead apply it to evaluate infinite (or finite) sums:

Determine the nodes $x_1, x_2$ and weights $w_1, w_2$ so that the formula

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} = w_1 f(x_1) + w_2 f(x_2)$$

becomes exact for polynomials $f(x)$ of as high degree as possible.

**Hint:** Sums of the form $\sum_{n=0}^{\infty} \frac{e^x}{n!}$ can be evaluated in closed form by considering derivatives of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ at $x = 1$. 