On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. Show all work in your bluebook. Textbooks, class notes and calculators are not permitted, although you are allowed to use a one-page reminder sheet.

Do problems 1, 2 and 3. Then, choose two of the three problems 4-6 on page 2. Indicate which problem you are skipping by putting an X through that number on your grading sheet.

Please sign your bluebook under the Honor Code to indicate that you have neither given nor received unauthorized assistance on this exam.

1. (40 points) Consider the set of equations $9x - 12y + 18z = 7$, $-3x + 4y - 6z = c$, where $c$ is some real number.
   (a) Write the system in matrix form $Ax = b$.
   (b) State the Fundamental Theorem of Linear Algebra.
   (c) Find the range and cokernel for $A$.
   (d) Define the Fredholm compatibility conditions (Fredholm alternative) for a general system $Ax = b$. Find a value of $c$ that satisfies these conditions for the system in (a).
   (e) For this $c$ value, find the general solution to (a). Write the solution as $x = w + z$, where $w \in \text{corng}(A)$ and $z \in \ker(A)$.
   (f) Of the solutions in (d), which has the smallest Euclidean norm?

2. (40 points) For each property given below, write down a matrix $A$ with that property or explain why no such matrix exists.
   (a) $\text{rng}(A) = \text{span}\left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$, and $\text{corng}(A) = \text{span}\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$.
   (b) $A = A^{-1}$.
   (c) The vector $\begin{pmatrix} 2 \\ -9 \\ 6 \end{pmatrix}$ is in the kernel of $A$, $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ is in the corange of $A$, and $\det A = 1$.
   (d) $A$ is real and symmetric with eigenvalues $4 + i$ and $4 - i$.
   (e) $A$ has an eigenvalue $3$ with multiplicity two, but only one eigenvector.

3. (30 points) Let $q(x, y, z) = x^2 + 4xy + 5y^2 + yz + z^2$ be a quadratic form.
   (a) Write $q = x^T K x$ for a matrix $K$.
   (b) Compute the $LU$ decomposition of $K$.
   (c) Define positive definite matrix. Is $q$ a positive definite quadratic form? Explain.
   (d) What do (a)-(c) allow you to conclude about the eigenvalues of $K$? (DO NOT COMPUTE the $\lambda$'s)
   (e) $\lambda_1 = 1$ is an eigenvalue of $K$. Find its associated eigenvector.
   (f) What is the sum of the other two eigenvalues, $\lambda_2 + \lambda_3$? Note: DO NOT compute $\lambda_2$ and $\lambda_3$ separately! Do not even think about wasting precious time doing this.
4. (20 points) Let $A$ be an $n \times n$ matrix such that every column of $A$ consists of only the number two, i.e. the $i^{th}$ column of $A$, say $A_i$, is of the form

$$A_i = \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}.$$

(a) Find the eigenvalues of $A$.
(b) Find the eigenvectors of $A$.
(c) What is $\text{dim}(\text{ker}(A))$?
(d) Explain why you know that you have found all possible eigenvalues and eigenvectors.

5. (20 points) General questions about matrices.

(a) Suppose the matrix $A$ is diagonalizable. Show that $A + 2I$ is also diagonalizable.
(b) Suppose $x$ and $y$ are both in the corange of some $m \times n$ matrix $A$ and that $Ax = Ay$. Show that $x - y$ is in the kernel of $A$. Does $x = y$?
(c) Suppose $A$ is a $n \times n$ diagonalizable matrix. Let

$$\sin(A) = \sum_{j=0}^{\infty} (-1)^j \frac{A^{2j+1}}{(2j + 1)!}.$$

Show that $\sin(A)$ is diagonalizable and find the eigenvalues of $\sin(A)$. Note, you need to use the fact that $\sin(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j + 1)!}$.

6. (20 points) Let $Q$ be the matrix

$$Q = I - 2 \frac{vv^T}{v^Tv}$$

where $v$ is an $n$-dimensional vector. This type of matrix is called a Householder reflection matrix.

(a) What is the size of the matrix $v^Tv$? Of the matrix $vv^T$?
(b) Suppose the vector $v = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$. Show that $Q^TQ = I$.
(c) For an arbitrary $v$, show that $Q = Q^T$.
(d) Show that $Q^TQ = I$ for any arbitrary $v$.

Have a great summer!
On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. **Show all work** in your bluebook. Start each problem on a **new page**. A correct answer with no supporting work may receive **no credit** while an incorrect answer with some correct work may receive partial credit. One page of notes is permitted, but no other books or electronic devices are allowed. **Sign your bluebook under the Honor Code to indicate that you have neither given nor received unauthorized assistance on this exam.**

Do problems 1, 2 and 3. Then, choose two of the four problems 4-6 on page 2. Indicate **which problem you are skipping** by putting an X through that number on your grading table.

1. (40 Points) For this problem, use the matrix and vector

   \[ A = \begin{pmatrix}
   0 & 1 & 0 \\
   0 & 0 & 1 \\
   1 & -4 & 4
   \end{pmatrix}, \quad b = \begin{pmatrix}
   1 \\
   1 \\
   1
   \end{pmatrix} \]

   (a) Is \( A \) regular or not? If it is regular, find its \( LU \) decomposition. If not, find its permuted \( LU \) decomposition.

   (b) Does \( Ax = b \) have a solution for every \( b \)? Why or why not? Using Gaussian elimination, find the solution for the given \( b \).

   (c) What is the characteristic polynomial \( p_A(\lambda) \) of \( A \)?

   (d) Show that \( \lambda = 1 \) is an eigenvalue of \( A \). What is the eigenvector for this \( \lambda \)?

   (e) Define a **complete** matrix. Is \( A \) complete?

2. (40 points) **State the fundamental theorem of linear algebra.** For each property given below, give an explicit matrix \( A \) with that property or explain why no such matrix exists.

   (a) \( \text{rng}(A) = \text{span}\left\{ \begin{pmatrix}
   2 \\
   1
   \end{pmatrix} \right\}, \text{ and corng}(A) = \text{span}\left\{ \begin{pmatrix}
   1 \\
   2 \\
   -1
   \end{pmatrix} \right\}. \]

   (b) The vector \( \begin{pmatrix}
   1 \\
   -1 \\
   2
   \end{pmatrix} \) is in the kernel of \( A \) and \( \begin{pmatrix}
   1 \\
   2 \\
   1
   \end{pmatrix} \) is in the corange of \( A \).

   (c) \( \ker A = \{0\} \) and \( \text{coker}(A) = \text{span}\left\{ \begin{pmatrix}
   1 \\
   2
   \end{pmatrix} \right\}. \]

   (d) \( A \) has an eigenvalue \( -1 \) with multiplicity two, but only one eigenvector.

   (e) \( A \) is real, symmetric, and has eigenvalues \( 2 + i \) and \( 2 - i \).

3. (30 points) Let \( S = \{ p(x) \in P^{(2)} \mid p(1) = 0 \} \), that is the space of quadratic polynomials that vanish at \( x = 1 \).

   (a) Define **vector subspace**.

   (b) **Show** that \( S \) is a vector subspace of \( P^{(2)} \).

   (c) Define the \( L^2 \) **inner product** on the interval \( (0, 2) \).

   (d) Find an orthogonal basis for \( S \) using the inner product in (c).

   (e) What is \( \dim(S) ? \) What is the dimension of \( S^\perp \), the orthogonal complement of \( S \) in \( P^{(2)} \)?
Do **TWO** of the following **THREE** problems.

Mark the problem that you skip with an X in your grading table.

4. (20 points) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

   (a) Define the **rank** of a matrix. What is $\text{rank}(A)$?
   (b) Find the singular values of $A$.
   (c) Find the singular value decomposition of $A$.
   (d) What is the **condition number** of a matrix? What is the condition number of $A$?

5. (20 points) Suppose the first row of a $2 \times 2$ real matrix $A$ is $(7, 5)$ and the eigenvalues of $A$ are $i$ and $-i$.

   (a) What is $\text{tr}(A)$?
   (b) What is $\text{det}(A)$?
   (c) Find $A$. Note: $A$ must be a real matrix!.

6. (20 points) Prove that if $\lambda_1 \neq \lambda_2$ are two distinct eigenvalues of a symmetric matrix $A$, then their corresponding eigenvectors, $v_1$ and $v_2$, are orthogonal using the Euclidean inner product (the dot product).

   Have a great summer and may the Fundamental Theorem be with you.
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Do problems 1, 2 and 3. Then, choose two of the three problems 4-6 on page 2. Indicate which problem you are skipping by putting an X through that number on your grading sheet.

Please sign your bluebook under the Honor Code to indicate that you have neither given nor received unauthorized assistance on this exam.

1. (30 points) Consider the set of equations $6x - 8y + 12z = 5$, $-3x + 4y - 6z = c$, where $c$ is some real number.
   
   (a) Write the system in matrix form $Ax = b$.
   
   (b) Define the fundamental subspaces range and cokernel for an arbitrary matrix. Find these spaces for $A$.
   
   (c) Define the Fredholm compatibility conditions (Fredholm alternative) for a general system $Ax = b$. Find a value of $c$ that satisfies these conditions for the system in (a).
   
   (d) For this $c$ value, find the general solution to (a). Write the solution as $x = w + z$, where $w \in \text{corng}(A)$ and $z \in \ker(A)$.
   
   (e) Of the solutions in (d), which has the smallest Euclidean norm?

2. (30 points) State the fundamental theorem of linear algebra For each property given below, write down a matrix $A$ with that property or explain why no such matrix exists.
   
   (a) $\text{rng}(A) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$, and $\text{corng}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.
   
   (b) $A = A^{-1}$.
   
   (c) The vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is in the kernel of the $A$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in the corange of $A$, and $\det A = 1$.
   
   (d) $A$ is real and symmetric with eigenvalues $1 + i$ and $1 - i$.
   
   (e) $A$ has an eigenvalue 2 with multiplicity two, but only one eigenvector.

3. (30 points) Let $q(x, y, z) = x^2 + 2xy + 3y^2 + 4yz + 4z^2$ be a quadratic form.
   
   (a) Write $q = x^T K x$ for a matrix $K$.
   
   (b) Compute the LU decomposition of $K$.
   
   (c) Define positive definite matrix. Is $q$ a positive definite quadratic form? Explain.
   
   (d) What do (a)-(c) allow you to conclude about the eigenvalues of $K$? (DO NOT COMPUTE the $\lambda$'s)
   
   (e) $\lambda_1 = 2$ is an eigenvalue of $K$. Find its associated eigenvector.
   
   (f) What is the sum of the other two eigenvalues, $\lambda_2 + \lambda_3$? Note: DO NOT compute $\lambda_2$ and $\lambda_3$ separately! Do not even think about wasting precious time doing this.

4. (20 points) Let $S = \text{span}\{ (x - 1), (x - 1)^2 \}$.
   
   (a) Define vector subspace.
(b) Show that $S$ is a vector subspace of $P^{(2)}$, the space of quadratic polynomials.
(c) What is $\dim(S)$?
(d) Define the $L^2$ inner product on the interval $(0, 2)$.
(e) Find the orthogonal complement $S^\perp$ to $S$ in $P^{(2)}$ using the inner product in (d).

5. (20 points) Let $L[x] = \begin{pmatrix} x_1 + 3x_2 \\ -x_1 + x_2 \\ 2x_2 \end{pmatrix}$ be a linear transformation.

(a) Which vector space is the domain of $L$? Which is the target space of $L$?
(b) Suppose $\{v_i : i = 1, \ldots, n\}$ is a basis for the domain of $L$. What is $n$? Why? (DO NOT FIND $v_i$.)
(c) If $L[v_1] = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$, what is $v_1$?
(d) Find a basis for the range of $L$.

6. (20 points) Let $Q$ be the matrix

$$Q = I - 2\frac{vv^T}{v^Tv}$$

where $v$ is an $n$-dimensional vector. This type of matrix is called a Householder reflection matrix.

(a) What is the size of the matrix $v^Tv$? Of the matrix $vv^T$?
(b) Suppose the vector $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Show that $Q^TQ = I$.
(c) For an arbitrary $v$, show that $Q = Q^T$.
(d) Show that $Q^TQ = I$ for any arbitrary $v$.

Have a great holiday and enjoy your spring semester!
On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. **Explain all of your answers.** This test is worth 150 points. There are 20 additional points available if you choose to answer all the questions. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. No books or notes. No electronic devices of any kind (e.g. cell phones, calculators, etc.) are permitted. Begin each problem on a new page.

1. (30 points) Let \( \mathbb{P}^{(4)} \) denote the vector space of all polynomials of degree less than or equal to 4.
   (a) Are \( p_1(x) = x - 2, p_2(x) = x^2 - 5x + 4, p_3(x) = 3x^2 - 4x, p_4(x) = x^2 - 1 \) linearly independent elements of \( \mathbb{P}^{(4)} \)?
   (b) What is the dimension of \( V = \text{span}\{p_1, p_2, p_3, p_4\} \)?
   (c) Verify the **Cauchy-Schwartz** inequality for the functions \( p_1 = x - 2 \) and \( p_4 = x^2 - 1 \) with respect to the \( L^2 \)-norm on \([0, 1]\).

2. (40 points) Let \( K = K^T \) be a symmetric \( n \times n \) matrix.
   (a) We learned that \( K \) is positive definite if and only if all of its eigenvalues are strictly positive. Prove the direction: \( K > 0 \) implies all of its eigenvalues are strictly positive.
   (b) Prove that \( K^2 \) is positive definite.
   (c) Prove that if \( K \) is positive definite then \( K \) can be written as a Gram matrix.
   (d) Find the Gram matrix \( K \) for the monomials 1, \( x, x^2 \) under the \( L^2 \) inner product on \([0, 1]\).

3. (50 points) For this problem let \( A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \).
   (a) Find the eigenvalues and eigenvectors of \( A \).
   (b) Is \( A \) positive definite, positive semi-definite or neither? Explain.
   (c) Find orthonormal eigenvector bases for each of the four fundamental subspaces of \( A \).
   (d) Find and orthonormal basis for \( \mathbb{R}^3 \) consisting of eigenvectors of \( A \). Verify orthonormality for full credit.
   (e) Write out the spectral factorization of \( A \).

4. (40 points) A few unrelated short answer questions.
   (a) State the Fundamental Theorem of Linear Algebra parts 1 and 2. (Hint: part 1 characterizes the dimensions of the four fundamental subspaces of a matrix; part 2 gives orthogonality relations between them.)
   (b) Give the definition for \( W \) to be a subspace of a vector space \( V \).
   (c) State how positive definite matrices are related to inner products on \( \mathbb{R}^n \).
   (d) Suppose the vector space \( V = \text{span}\{v_1, v_2, \ldots, v_n\} \). What do you know about \( \text{dim}(V) \)?

5. (10 points) For the quadratic form \( q(x) = 2x_1^2 + x_1x_2 - 2x_1x_3 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \), find the vector \( v \in \mathbb{R}^3 \) that minimizes \( q(x) \). (Hint: use a matrix to represent \( q(x) \).)
1. (20 points) Let \( P^{(3)} \) denote the vector space of all polynomials of degree less than or equal to 3.

(a) Are \( q_1(x) = x^2 + 1 \), \( q_2(x) = x^2 - 1 \) and \( q_3(x) = 7 \) linearly independent elements of \( P^{(3)} \)?

(b) What is the dimension of \( V = \text{span}\{q_1, q_2, q_3\} \)?

(c) Verify the Triangle Inequality for the functions \( q_1(x) = x^2 + 1 \) and \( q_2(x) = x^2 - 1 \) with respect to the \( L^2 \)-norm on \([0,1]\).

2. (20 points)

(a) Let \( V \) be an inner product space. State what it means for an \( n \times n \) matrix \( K \) to be the Gramm matrix associated to vectors \( v_1, \ldots, v_n \in V \).

(b) Prove that every positive definite \( n \times n \) matrix \( K \) can be written as a Gramm matrix.

(c) Prove that a positive definite \( n \times n \) matrix \( K \) has positive trace: \( \text{tr}(K) > 0 \).

3. (30 points)

(a) Let \( P \) denote the orthogonal projection of \( \mathbb{R}^3 \) onto the plane \( V = \{ z = x + y \} \). Find the matrix representation of \( P \).

(b) Let \( Q \) denote the orthogonal projection of \( \mathbb{R}^3 \) onto the plane \( W = \{ z = x - y \} \). Find the matrix representation of \( Q \).

(c) Is the composition \( R = Q \circ P \) the same as the orthogonal projection of \( \mathbb{R}^3 \) onto \( L = V \cap W \)? Why or why not? Justify your answer.

4. (50 points)

(a) State the Fundamental Theorem of Linear Algebra. Be sure to include the hypothesis of the theorem in your answer.

(b) Let \( A \) be a real \( m \times n \) matrix. Show that the linear system \( Ax = b \) has a solution if and only if \( b \) is orthogonal to \( \text{coker}(A) \). (Hint: If \( W \) is a finite-dimensional subspace of an inner product space, then \( (W^\perp)^\perp = W \).)

(c) Find the dimensions of the four fundamental subspaces associated to \( A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \).

(d) Find a basis for \( \ker(A) \) and \( \text{rng}(A) \).

(e) Find an orthonormal basis for \( \text{rng}(A) \).

(f) Find a QR factorization of \( A \).

PLEASE TURN OVER. MORE PROBLEMS ON THE OTHER SIDE.
5. (30 points) For this problem let \( A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \).

(a) Find the eigenvectors and eigenvectors of the matrix \( A \).

(b) Is \( A \) positive definite? Why or why not?

(c) Find an orthonormal eigenvector basis of \( \mathbb{R}^3 \) determined by \( A \) or explain why none exists.

(d) Write out the spectral factorization of \( A \) if possible.

6. (Extra Credit - 10 point bonanza)

Find the closest point on the plane spanned by \( (1, 1, 0, 0)^T \) and \( (0, 0, 1, 1)^T \) to \( b = (3, 1, 2, 1)^T \). What is the distance between \( b \) and the plane?
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table. Show all work in your bluebook. Textbooks, class notes and calculators are not permitted,
although you are allowed to use one page of notes as a reminder sheet. If you find that the arithmetic
for this exam seems complicated, go back and check your work.

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Please skip one of the 30 point problems, #2, 4, or 5. Write the number of the skipped
problem on the front of your bluebook.

1. (50 points) Each of the following unrelated questions involves a concept from class. The concept
is in bold-face. A complete answer for each part will include a definition of the bold-faced word.
Then, use your definition in your answer to the question.

(a) Are the 4 matrices given by
\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},
\]
a basis of \( M_{2 \times 2} \)?

(b) What are the \( L^1 \), \( L^2 \), and \( L^\infty \) norms on \([-1,1]\) for the function \( f(x) = x - (1/4) \)?

(c) Find the rank of the \( m \times n \) matrix \( A = vw^T \), where \( v \) is a nonzero \( m \times 1 \) vector and \( w^T \) is
a nonzero \( 1 \times n \) row vector.

(d) Is the function \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( F \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} xy \\ x - y \end{array} \right) \) a linear function? Explain.

2. (30 points) For this problem, let \( A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} \).

(a) Find \( \text{corng}(A) \) and \( \ker(A) \).

(b) Find conditions on vector \( b \) so that \( Ax = b \) has a solution.

(c) For \( b = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \) find the general solution \( Ax = b \) given by \( x = w + z \) with \( w \in \text{corng}(A) \) and
\( z \in \ker(A) \).

3. (35 points) As you know, an \( n \times n \) matrix is symmetric if \( A = A^T \). An \( n \times n \) matrix \( J \) is called
skew-symmetric if \( J = -J^T \). (Skew-symmetric matrices arise in many applications, particularly
in physics.)

(a) Suppose a matrix \( J \) is skew-symmetric. What can you say about the diagonal entries of \( J, \)
\( J_{ii} \) for \( i = 1, \ldots, n \)? Explain.

(b) Write down an example of a \( 2 \times 2 \), non-zero, skew-symmetric matrix.

(c) Does the set of \( 2 \times 2 \) skew-symmetric matrices form a subspace of the vector space of all
\( 2 \times 2 \) matrices, \( M_{2 \times 2} \)? Explain fully, using the definition of a subspace in your answer.

(d) For your example in part (b) show that \( v^T J v = 0 \) for all \( v \in \mathbb{R}^2 \).

(e) Show that \( v^T J v = 0 \) for all \( v \in \mathbb{R}^n \), for any \( n \times n \), skew-symmetric matrix \( J \).
4. (30 points) For this problem, assume $C$ is a $3 \times 3$ matrix with eigenvalues 0, 1 and 2. Five of the following six quantities can be computed from this information. Determine which five and find their values. In each case, briefly explain your reasoning.

(a) the trace of $C$
(b) the rank of $C$
(c) the determinant of $C^T C$
(d) the eigenvalues of $C^T C$ (Hint: What do the eigenvalues of $C^T C$ represent?)
(e) the eigenvalues of $(C + I)^{-1}$ (Hint: What does the characteristic polynomial for $C + I$ tell you?)
(f) the eigenvalues of $C^3$.

5. (30 points) Let $D = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$.

(a) Give the definition for a matrix to be diagonalizable.
(b) Find the diagonalization for matrix $D$.
(c) Use part (b) to find $D^5$. (Use part (b), no credit if you just multiply $D$ by itself 5 times!)
(d) The most reasonable way to define the square root of a matrix, $A^{1/2}$, is to define $A^{1/2} = B$ when $A = B^2$. Use your answer in part (b) to find $D^{1/2}$.

6. (35 points) Let $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(a) Compute the singular value decomposition of $A$.
(b) Explain why the first $r$ columns of $U$ (or the columns of matrix $P$ if you are using our text’s formulation of SVD) give a basis for range($A$).
(c) Explain why the first $r$ columns of $V$ (or the columns of matrix $Q$) form a basis for corng($A$).
(d) Use the singular value decomposition of $A$ to write $A$ as the sum of two rank one matrices.

Extra Credit (up to 10 points). What is the most important thing (or useful to your major) you’ve learned this semester in this class?
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1. (20 points) Here are some short answer questions. As always, explain your answer!
   
   (a) Suppose $A$ is a $3 \times 3$ matrix with diagonal elements $1, 3,$ and $-4,$ and that two of its eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3.$ What is the third eigenvalue?
   
   (b) If $A$ is a square matrix and $A = 5B,$ then the eigenvalues of $A$ are five times the eigenvalues of $B.$ True or False?
   
   (c) Prove that the product of the singular values of a square, nonsingular matrix $A$ is equal to $|\det A|.$
   
   (d) Suppose a $3 \times 3$ matrix $A$ has eigenvalues $\lambda = 1, 2,$ and $3.$ What are the eigenvalues of $B = A - 2I$?

2. (30 points) Consider the system $Ax = b,$ with $A = \begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

   (a) Find a basis for $\text{rng}(A)$.
   
   (b) Find a basis for $\text{coker}(A)$. Explain how your calculation is consistent with the fundamental theorem of Linear Algebra.
   
   (c) Does the system $Ax = b$ have a solution? Why or why not?
   
   (d) Find the least squares solution to $Ax = b$.

3. (20 Points) A few True-False questions. You **must** explain your answer!

   (a) The quadratic form $q(x, y) = x^2 + 2xy + 3y^2$ is positive definite.
   
   (b) The expression $(\mathbf{v}, \mathbf{w}) = v_1 w_1 + 2v_1 w_2 + 3w_2 v_2$ is an inner product on $\mathbb{R}^2$.
   
   (c) If $AB = I,$ then $BA = I$.
   
   (d) If $A$ and $B$ are invertible, then so is $A + B$.
   
   (e) If $\mathbf{v}$ and $\mathbf{w}$ are nonzero column vectors in $\mathbb{R}^n,$ then $\text{rank}(\mathbf{v}\mathbf{w}^T) = 1.$
4. (20 points). Let \( \mathcal{V} \) be the set of all polynomials \( p(x) \) of at most cubic order with real coefficients such that \( p(1) = 0 \).

(a) Show that \( \mathcal{V} \) is a vector space.

(b) Find a basis \( p_i(x), i = 1, \ldots \) for \( \mathcal{V} \). What is the dimension of \( \mathcal{V} \)?

(c) Using the \( L^2 \) inner product on the interval \([-1, 1]\), find the inner product of your basis vectors \( p_1 \) and \( p_2 \).

(d) Use the Gram-Schmidt procedure to find a new basis vector \( q_2(x) \) that is orthogonal to \( p_1(x) \).

5. (30 points) For this problem let \( A = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix} \).

(a) Find the \( LU \) decomposition of \( A \).

(b) Find the eigenvalues and eigenvectors of \( A \).

(c) “Complete matrices” can be diagonalized. Explain what this statement means. Is \( A \) complete?

(d) Compute \( A^5 \) (Hint: you should use the results you obtained above; no credit if you just multiply \( A \) by itself five times!).

6. (30 Points). Using the same matrix \( A \) as the previous problem

(a) Find the singular values of \( A \).

(b) What is the condition number of \( A \)?

(c) Find the singular value decomposition of \( A \).

(d) Find the pseudoinverse \( A^+ \) of \( A \).

EXTRA CREDIT (20 Points) A fundamental theorem that we did not cover in class is the Cayley-Hamilton Theorem which says that “a square matrix is a solution of its own characteristic equation.”

1. Let \( p(\lambda) \) be the characteristic equation for a matrix \( A \). Write out the form of this equation in terms of the eigenvalues of \( A \). The Cayley-Hamilton theorem states that \( p(A) = 0 \). Explain what this equation means.

2. Suppose that \( A \) is a diagonal matrix. Prove that \( p(A) = 0 \).

3. Suppose that \( A \) is a complete matrix. Use analysis like that in Problem 5 (d) to prove that \( p(A) = 0 \).

4. It is much harder to show that this result works for incomplete matrices as well. I think you’ve worked hard enough for now. Enjoy your break!
1. Consider the system $Bx = b$, where $B = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$. (a) Find the set of all possible least-squares solutions of this system.

(5 points) Solve $B^T B \hat{x} = B^T b$, and $BB^T = \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$, $B^T b = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$

so $\hat{x}_1 = 3$, $\hat{x}_2 = 1/2$

The solution is $\hat{x} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$. Two points off if you rescale the solution, four points off if you think $\hat{b}$ is the solution.

(b) Find the projection of $b$ onto the column space of $B$.

(3 points) The quickest way to solve is to note that $\hat{b} = B\hat{x} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ is the projection of $b$ onto the column space of $B$. You could also note that the columns of $B$ are orthogonal and compute $\hat{b}$ by orthogonal projection. (2 points off if you only project onto one column of $B$).

(c) Find the least-squares error.

(3 points) The error is $||b - \hat{b}||$. The vector $b - \hat{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so $||b - \hat{b}|| = \sqrt{2}$. Two points off if you rescaled a vector or didn’t find the length of the vector.

2. You are given a matrix $A$ and three linearly independent vectors $u, v, w$. The matrix $A$ has only two distinct eigenvalues, the eigenspace of $\lambda_1$ is $\text{Span}\{u, v\}$, and the eigenspace of $\lambda_2$ is $\text{Span}\{w\}$.

(a) Is it possible that $A$ is a $2 \times 2$ matrix? If so, is it diagonalizable? (5 points)

$A$ cannot be a $2 \times 2$ matrix because it is impossible to have 3 linearly independent vectors in $\mathbb{R}^2$.

(b) Is it possible that $A$ is a $3 \times 3$ matrix? If so, is it diagonalizable?

$A$ could be a $3 \times 3$ matrix, because 3 linearly independent vectors in $\mathbb{R}^3$ are possible. To diagonalize $A$ we would need 3 linearly independent eigenvectors, which we have. So $A$ is diagonalizable.

(c) Is it possible that $A$ is a $4 \times 4$ matrix? If so, is it diagonalizable?

$A$ could be $4 \times 4$, but it wouldn’t be diagonalizable because it doesn’t have 4 linearly independent eigenvectors.

3. Consider $A = \frac{1}{4} \begin{bmatrix} -3 & 5 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -3/4 & 5/4 \\ 5/4 & -3/4 \end{bmatrix}$. Use this definition of $A$ for all parts of this problem.

(a) What is $A^T$? (2 points)
$A^T = \begin{bmatrix} -3/4 & 5/4 \\ 5/4 & -3/4 \end{bmatrix} = A$. The point of this part of the problem is to recognize that $A$ is symmetric, and thus is diagonalizable with real eigenvalues and orthogonal eigenvectors.

(b) Calculate the eigenvalues of $A$. (3 points)

$$\det (A - \lambda I) = \begin{vmatrix} -3 - \lambda & 5 \\ 5 & -3 - \lambda \end{vmatrix} = \frac{1}{4} (\lambda^2 + 6\lambda - 16) = \frac{1}{4} (\lambda + 8)(\lambda - 2) = 0.$$ So the eigenvalues are $\lambda_1 = -8$ and $\lambda_2 = 2$.

(c) Calculate the corresponding eigenvectors. (5 points)

$$(A - \lambda_1 I)x_1 = \frac{1}{4} \begin{vmatrix} 5 & 5 \\ 5 & 5 \end{vmatrix} x_1 = 0.$$ Therefore $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

And $(A - \lambda_2 I)x_2 = \frac{1}{4} \begin{vmatrix} -5 & 5 \\ 5 & -5 \end{vmatrix} x_2 = 0$. Therefore $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(d) Are these eigenvectors linearly independent? Why? (3 points)

Yes, they are linearly independent. Eigenvectors corresponding to distinct eigenvalues are always linearly independent.

(e) Are these eigenvectors orthogonal? If not, make them orthogonal. (3 points)

Yes, they are orthogonal, which we can check by noting that $x_1^T x_2 = 0$. They must be orthogonal, since the spectral theorem says that for a real symmetric matrix, eigenvectors corresponding to distinct eigenvalues are orthogonal.

(f) Are these eigenvectors orthonormal? If not, normalize them. (3 points)

In the form given above, they are not normalized. The normalized versions are $x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(g) Are these eigenvectors a basis for some vector space? If not, why not? If so, what vector space? (3 points)

These eigenvectors are a basis for $\mathbb{R}^2$, since they are linearly independent and span the space.

(h) Consider the dynamical system $x_k = Ax_{k-1}$. What is the limiting behavior of $x_k$ as $k \to \infty$ for $x_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$? (3 points)

The initial condition $x_0$ is the eigenvector for $\lambda_2 = 2$. Therefore the solution for all $k$ is $x_k = 2^k x_0 = 2^k \begin{bmatrix} 5 \\ 5 \end{bmatrix}$. This solution $\to \infty$ for large $k$ (unstable).

(i) Consider the dynamical system $x_k = Ax_{k-1}$. What is the limiting behavior of $x_k$ as $k \to \infty$ for $x_0 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$? (3 points)

The initial condition $x_0$ is the eigenvector for $\lambda_1 = -8$. Therefore the solution for all $k$ is
\[ x_k = (-8)^k x_0 = (-8)^k \begin{bmatrix} -3 \\ 3 \end{bmatrix}. \] This solution \( \to \infty \) for large \( k \) (unstable), although it changes direction every step.

(j) Consider the dynamical system \( \frac{dx}{dt} = Ax \). What is the limiting behavior of \( x(t) \) as \( t \to \infty \) for \( x_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \)? (3 points)

As in part (h), the initial condition \( x_0 \) is the eigenvector for \( \lambda_2 = 2 \). Therefore the solution for all time is \( x(t) = e^{2t}x_0 = e^{2t} \begin{bmatrix} 5 \\ 5 \end{bmatrix} \). This solution \( \to \infty \) for large \( t \) (unstable).

(k) Consider the dynamical system \( \frac{dx}{dt} = Ax \). What is the limiting behavior of \( x(t) \) as \( t \to \infty \) for \( x_0 = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \)? (3 points)

As in part (i), the initial condition \( x_0 \) is the eigenvector for \( \lambda_1 = -8 \). Therefore the solution for all time is \( x(t) = e^{-8t}x_0 = e^{-8t} \begin{bmatrix} -3 \\ 3 \end{bmatrix} \). This solution \( \to 0 \) for large \( t \) (stable).

(l) Find matrices \( S, \Lambda, \) and \( S^{-1} \) such that \( A = SAS^{-1} \). (3 points)

Use the eigenvalues and eigenvectors from above. The matrix \( S = Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \) with \( S^{-1} = Q^T \), and \( \Lambda = \begin{bmatrix} -8 & 0 \\ 0 & 2 \end{bmatrix} \).

4. In this problem you will work with the space of \( 2 \times 2 \) matrices \( \mathbb{M}_{2 \times 2} \). You may take as given that \( \mathbb{M}_{2 \times 2} \) is a vector space, under the usual operations of addition of matrices and multiplication by real scalars. (a) Is the set of all rank-one \( 2 \times 2 \) matrices a subspace of \( \mathbb{M}_{2 \times 2} \)? Why or why not? (5 points)

It is not a subspace because it doesn’t include the zero matrix. (The zero matrix is not a rank one matrix.)

(b) Show that the set of all upper triangular \( 2 \times 2 \) matrices is a subspace of \( \mathbb{M}_{2 \times 2} \). Call this subspace \( V \).

The set of all upper triangular \( 2 \times 2 \) matrices is a subset of \( \mathbb{M}_{2 \times 2} \). A general upper triangular matrix has the form \( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \). Thus the zero matrix is in \( V \) if we choose \( a = b = c = 0 \). If we add, \( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix} \), the result is still upper triangular, so \( V \) is closed under addition. \( V \) is also closed under scalar multiplication, because \( r \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ra & rb \\ 0 & rc \end{bmatrix} \) is in \( V \). Therefore \( V \) is a subspace of \( \mathbb{M}_{2 \times 2} \).

(c) Find a basis for the subspace \( V \).

A basis is \( \{ b_1, b_2, b_3 \} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \). By adding these 3 matrices together...
with different weights, we could form any 2 × 2 upper triangular matrix.

(d) What is the dimension of the subspace $V$? Explain how this is related to the number of entries in a 2 × 2 upper triangular matrix that can be chosen independently.

The dimension of $V$ is the number of basis vectors, which is 3. This is the same as the number of entries that can be chosen independently, because the entries one can choose are exactly the weights of the linear combination of basis vectors.

(e) Consider the transformation $T : V \rightarrow V$ which maps a 2 × 2 upper triangular matrix $U$ to $AU$, where $A = \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$. Determine how your basis vectors from part (c) transform under $T$.

\[
A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 4 \end{bmatrix}
\]

(f) Find the matrix of the transformation $T$ relative to your basis from part (c).

We can describe an arbitrary 2 × 2 upper triangular matrix as a linear combination of the basis vectors: \[
\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \mathbf{b}_1 + b \mathbf{b}_2 + c \mathbf{b}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_V.
\]

In other words, the matrix can be represented in the basis $V$ by the coordinates $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. In part (e), you found that under the transformation $b_1 \rightarrow -1 b_1$, or $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. The transformed vector is the first column of the matrix. Since $b_2 \rightarrow -1 b_2$ and $b_3 \rightarrow 3 b_2 + 4 b_3$, the other columns are $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. Thus the matrix of the transformation is $[T]_V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

6. Consider two nonzero vectors $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{v} \in \mathbb{R}^n$. (a) Use these vectors to construct an $m \times n$ matrix $M$.

$M = \mathbf{uv}^T$

(b) Find a basis for the column space of $M$.

Every column is a multiple of $\mathbf{u}$, therefore $\mathbf{u}$ is a basis of the column space.

(c) Find a basis for the row space of $M$.

Every row is a multiple of $\mathbf{v}$, therefore $\mathbf{v}$ is a basis for the row space.

(d) What is the rank of $M$?

The rank is one, because there is one linearly independent column.

(e) Find an eigenvector of $M$ and an expression for the eigenvalue (when $m = n$).
When \( m = n \), then \( \mathbf{u} \) is an eigenvector because \( \mathbf{u} \mathbf{v}^T \mathbf{u} = (\mathbf{v}^T \mathbf{u}) \mathbf{u} \), and the eigenvalue is \( \mathbf{v}^T \mathbf{u} \).

6. The singular value decomposition of the matrix \( A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \) is

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

(a) Find the pseudo-inverse of \( A \). (5 points)

The pseudo-inverse is \( A^+ = Q_2 \Sigma^+ Q_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A^T. \)

(b) Find the minimum length least-squares solution to \( A \mathbf{x} = \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \). (3 points)

The solution is \( \mathbf{x}^+ = A^+ \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \).

(c) The minimum length least-squares solution \( \mathbf{x}^+ \) is in one of the four fundamental subspaces of \( A \). Which one? Explain. (3 points)

The solution \( \mathbf{x}^+ \) is in the row space of \( A \), by construction: the minimum length least-squares solution is always in the row space.

(d) The vector \( A \mathbf{x}^+ \) is in one of the four fundamental subspaces of \( A \). Which one? Explain. (3 points)

When we multiply a matrix \( A \) times any vector, the result is always in the column space of \( A \).

Therefore \( A \mathbf{x}^+ = AA^+ \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) is in the column space of \( A \). Note that in this case, we just recovered the original vector \( \mathbf{b} \).

(e) Find the projection of \( \mathbf{b} \) onto the column space of \( A \). (Hint: the answers to part (c) and (d) should help you. 3 points)

Since \( A \mathbf{x}^+ = AA^+ \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) is in the column space of \( A \), the projection onto the column space is just \( \mathbf{b} \)—the vector is already in the column space of \( A \).

(f) Find the projection of \( \mathbf{b} \) onto the left nullspace of \( A \). (3 points)

The left nullspace component in general is \( AA^+ \mathbf{b} - \mathbf{b} \), which here is zero.

7. Proofs.

(a) If \( A \) and \( B \) are similar matrices, prove that \( \det A = \det B \). (5 points)

If \( A \) and \( B \) are similar, then we can write \( A = MBM^{-1} \) for some matrix \( M \). Then

\[
\det A = \det MBM^{-1},
\]

\[
= \det M \det B \det M^{-1},
\]
$= \frac{\det M \det B}{\det M}$,
$= \det B.$

(b) If $A$ is a real, symmetric matrix, prove that eigenvectors corresponding to distinct eigenvalues are orthogonal (without recourse to the spectral theorem). (10 points)

We have two eigenvectors of distinct eigenvalues: $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$ with $\lambda_1 \neq \lambda_2$. Take the inner product of $x_1$ with $Ax_2$:

$$x_1^T A x_2 = x_1^T \lambda_2 x_2,$$
$$= \lambda_2 x_1^T x_2.$$

It is also true that

$$x_1^T A x_2 = x_1^T A^T x_2,$$
$$= (Ax_1)^T x_2,$$
$$= (\lambda_1 x_1)^T x_2,$$
$$= \lambda_1 x_1^T x_2.$$

This proves that $\lambda_1 x_1^T x_2 = \lambda_2 x_1^T x_2$. Now for this equation to be true, there are only two options: (a) we could have $\lambda_1 = \lambda_2$, but this is impossible, since we initially assumed that the eigenvalues are distinct. The only other option is (b) $x_1^T x_2 = 0$, implying that the eigenvectors are orthogonal.

Good luck on the rest of your finals, and have a great summer.
Solution: APMM 3310: Matrix Methods — Final — Summer 2012

Problem 1. (60 points) Suppose $F$ is a $3 \times 4$ real matrix with
\[
\text{corange}(F) = \text{span} \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}, \quad \text{range}(F) = \text{span} \begin{bmatrix} 2 & 0 \\ 5 & 1 \\ 1 & -5 \end{bmatrix}, \quad \text{and} \quad F \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 0 \end{pmatrix}
\]

(a) What is the Fredholm Alternative for the matrix $F$ in terms of the cokernel of $F$?

(b) What is the Euclidean distance of the vector $b = (-15, 0, 0)^T$ from range$(F)$? (Use only the methods of this class to answer this question.)

(c) Find a solution of minimum Euclidean norm of the equation $Fx = (5, 13, 0)^T$.

(d) Find a general solution of the linear system $Fx = (5, 13, 0)^T$.

(e) Show that range$(F) \subset \text{range}(FF^T)$.

(f) Is $FF^T$ positive definite? Is $FF^T$ complete? How many nonnegative eigenvalues does $FF^T$ have, if any? How many nonzero eigenvalues does $FF^T$ have, if any? Justify your answers.

Solution:

(a) Note that $\text{coker}(F) \perp \text{range}(F)$ and so we need
\[
\begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
which implies
\[
\text{coker}(F) = \text{span} \begin{bmatrix} -13 \\ 5 \\ 1 \end{bmatrix}, \quad \text{so the compatibility condition is} \quad \begin{pmatrix} -13 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0.
\]

(b) Note that the projection of $b$ onto $\text{coker}(F)$ is
\[
z = \frac{(-13, 5, 1)^T \cdot (-15, 0, 0)^T}{195} \begin{pmatrix} -13 \\ 5 \\ 1 \end{pmatrix} = \frac{195}{195} \begin{pmatrix} -13 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -13 \\ 5 \\ 1 \end{pmatrix}
\]
and so the distance is $\|z\| = \sqrt{13^2 + 5^2 + 1} = \sqrt{195}$

Alternate method:

Note that the projection of $b$ onto $\text{range}(F)$ is
\[
w = \frac{(2, 5, 1)^T \cdot (-15, 0, 0)^T}{30} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \frac{(0, 1, -5)^T \cdot (-15, 0, 0)^T}{26} \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} = -1 \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}
\]
now note that $b = w + z$ where $z \in \text{range}(F)^\perp$ and so $z = b - w = \begin{pmatrix} -15 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -5 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 5 \\ 1 \end{pmatrix}$
and so the distance is $\|z\| = \sqrt{13^2 + 5^2 + 1} = \sqrt{195}$
Problem 2. (50 points) The following problems are unrelated:

(a) Suppose a 2×2 matrix $F$ has eigenvalues -1, and 3 with corresponding eigenvectors \( \begin{pmatrix} -1 \\ 2 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

Write down the matrix form of the linear transformation $L[u] = Fu$ in terms of the basis \( \begin{pmatrix} 9 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 3 \end{pmatrix} \).

(b) (10 pts) Suppose $M$ is an $n \times n$ matrix such that $M^2 = M$, find all possible eigenvalues of $M$.

(c) (15 pts) Find a spectral factorization of $K = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$.

(d) (10 pts) Let $B$ be an orthogonal $n \times n$ matrix, show that if $\lambda$ is an eigenvalue of $B$ then so is $1/\lambda$.

Solution: (a) Note that

\[
F = SAS^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} = \begin{pmatrix} -1 & 1/3 \\ 2 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & -1/3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5/3 & 4/3 \\ 8/3 & 1/3 \end{pmatrix}
\]

and

\[
B = T^{-1}FT = \begin{pmatrix} 1/9 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 5/3 & 4/3 \\ 8/3 & 1/3 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5/3 & 4/9 \\ 8 & 1/3 \end{pmatrix}
\]

(b) If $Mx = \lambda x$ then $M^2 x = M \lambda x = \lambda^2 x$ but $M^2 = M$ so we have $Mx = \lambda^2 x$ and thus $\lambda x = \lambda^2 x$ which implies $\lambda x - \lambda^2 x = (\lambda - \lambda^2)x = 0$ and thus $\lambda - \lambda^2 = 0$ and so $\lambda = 0, 1$
so we have

\[
\det\begin{pmatrix}
3 - \lambda & -1 & -1 \\
-1 & 2 - \lambda & 0 \\
-1 & 0 & 2 - \lambda
\end{pmatrix} = (3 - \lambda)(2 - \lambda)^2 - (-1)(-1)(2 - \lambda) + (-1)(2 - \lambda)
\]

\[
= (2 - \lambda) [(3 - \lambda)(2 - \lambda) - 1 - 1]
\]

\[
= (2 - \lambda)(\lambda^2 - 5\lambda + 4) = (2 - \lambda)(\lambda - 1)(\lambda - 4)
\]

so \(\lambda = 1, 2, 4\). Now we find the eigenvectors. If \(\lambda = 1\) we have

\[
[M - I|0] = \begin{pmatrix}
2 & -1 & -1 & 0 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix} 1 \\
1 \\
1
\end{pmatrix} z
\]

If \(\lambda = 2\) we have

\[
[M - 2I|0] = \begin{pmatrix}
1 & -1 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix} 0 \\
-1 \\
1
\end{pmatrix} z
\]

If \(\lambda = 4\) we have

\[
[M - 4I|0] = \begin{pmatrix}
-1 & -1 & -1 & 0 \\
-1 & -2 & 0 & 0 \\
-1 & 0 & -2 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix} -2 \\
1 \\
1
\end{pmatrix} z
\]

so we have

\[
M = QAQ^T = \begin{pmatrix}
1/\sqrt{3} & 0 & -2/\sqrt{6} \\
1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\
1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 4
\end{pmatrix}
\begin{pmatrix}
1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\
0 & -1/\sqrt{2} & 1/\sqrt{2} \\
-2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6}
\end{pmatrix}
\]

(d) If \(Bx = \lambda x\) then \(x = \lambda B^T x\) and so \(B^T x = \frac{1}{\lambda} x\) so \(\frac{1}{\lambda}\) is an eigenvalue of \(B^T\), but a matrix and its transpose have the same eigenvalues (since \(\text{det}(B - \lambda I) = \text{det}(B^T - \lambda I)\)), so \(\frac{1}{\lambda}\) is an eigenvalue of \(B\).

**Problem 3.** (40 points) The following problems are unrelated, **justify your answers:**

(a) Let \(F\) be a \(2 \times 2\) matrix with \(\text{det}(F) = 6\) and \(\text{tr}(F) = 5\), find all possible eigenvalues of \(F\).

(b) Let \(M = -M^T\) be a real skew-symmetric \(n \times n\) matrix, find all real eigenvalues of \(M\).

(c) Let \(K = -K^T\) be a real skew-symmetric \(n \times n\) matrix, is \(Y = I - K\) is invertible?

(d) Let \(B = -B^T\) be a real skew-symmetric \(n \times n\) matrix, is \(S = (I + B)(I - B)\) positive definite?

**Solution:**

(a) So we have \(\lambda_1 \lambda_2 = 6\) and \(\lambda_1 + \lambda_2 = 5\), which implies \(\lambda_1(5 - \lambda_1) = 6\) and so we have \(\lambda_1^2 - 5\lambda_1 + 6 = 0\) and thus \((\lambda_1 - 2)(\lambda_1 - 3) = 0\) and so if \(\lambda_1 = 2\) then \(\lambda_2 = 3\) and if \(\lambda_1 = 3\) then \(\lambda_2 = 2\).

(b) The only possible real eigenvalue is 0. Suppose there exists an eigenvalue \(\lambda\) and \(v \neq 0\) is the corresponding eigenvector, i.e., \(Mv = \lambda v\). Then \(v^T M v = \lambda v^T v = \lambda \|v\|^2\). Now note that since \(v^T M v\) is a scalar we have \(v^T M v = (v^T M v)^T = v^T M^T v = -v^T M v = -\lambda v^T v = -\lambda \|v\|^2\), so \(\lambda \|v\|^2 = -\lambda \|v\|^2\). Thus \(\lambda\) must be zero since \(\|v\|^2 \neq 0\).

(c) Yes, we show \(Y\) is invertible by showing \(\text{ker}(Y) = \ker(I - K) = \{0\}\). Consider \((I - K)x = 0\) then this implies \(x - Kx = 0\) and multiplying on the left by \(x^T\) yields \(x^T x - x^T K x = 0\) and so \(x^T x = x^T K x\), and note that since \(x^T K x\) is a scalar, we have that \(x^T K x = (x^T K x)^T = x^T K^T x = -x^T K x\) and so \(x^T K x = 0\), which implies \(x^T x = 0\). But note that \(x^T x = \|x\|^2\) and so we see that \(\|x\|^2 = 0\) which implies \(x = 0\). Thus we have that \(\text{ker}(Y) = \{0\}\) and so \(Y\) is invertible.
(d) For any $x \neq 0$, $x^T S x = x^T (I + B)(I - B)x = x^T (I - B^2)x$, now note that $-B^2 = -BB = B^T B$ and so $x^T S x = x^T (I - B^2)x = x^T (I + B^T B)x = x^T x + x^T B^T Bx$ and recall $x^T x = \|x\|^2 > 0$ and $x^T B^T Bx = \|Bx\|^2 \geq 0$. Thus $x^T S x = x^T x + x^T B^T Bx = \|x\|^2 + \|Bx\|^2 > 0$, so $S$ is positive definite.
Problem 1. (20 points) Consider the matrix \( F = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 3 & 2 \end{bmatrix} \).

(a) (10 pts) Find the LU-decomposition of \( F \).

(b) (5 pts) What is the rank of \( F \)?

(c) (5 pts) Is \( F \) invertible? Why or why not?

Solution:

(a) Here we have \( U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) and \( L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \).

(b) \( \text{rank}(F) = \text{rank}(U) = 3 \)

(c) \( \det(F) = \det(U) = -1 \cdot -3 \cdot 2 = 6 \neq 0 \), so \( F \) is invertible.

Problem 2. (10 points) Find the least squares solution to the linear system:

\[
\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}.
\]

Solution:

Note that \( K = A^T A = \begin{bmatrix} 14 & 13 \\ 13 & 26 \end{bmatrix} \) and \( f = A^T b = \begin{bmatrix} 28 \\ 32 \end{bmatrix} \) and so solving \( Kx = f \) yields \( x = \begin{bmatrix} 8/5 \\ 28/65 \end{bmatrix} \).

Problem 3. (20 points)

(a) (10 pts) Use the Gram-Schmidt process to construct an orthogonal basis of the plane \( P \) spanned by the vectors \( \{w_1, w_2\} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \).

(b) (10 pts) Find the orthogonal projection of the vector \( v = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \) onto the plane \( P \) from part (a).

Solution:

(a) \( u_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \) - \( \frac{w_2 \cdot u_1}{\|u_1\|^2} u_1 = \begin{bmatrix} 3/2 \\ 2 \\ -5/2 \end{bmatrix} \)

(b) \( \text{proj}(v) = c_1 u_1 + c_2 u_2 \) where \( c_i = \frac{v \cdot u_i}{\|u_i\|^2} \), so \( \text{proj}(v) = \begin{bmatrix} 8/15 \\ 44/15 \\ -4/3 \end{bmatrix} \)

Problem 4. (20 points) Consider the matrix \( M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \).

(a) (4 pts) Find the eigenvalues of the matrix \( M \).
(b) (6 pts) Find the corresponding eigenvectors of the matrix M.
(c) (2 pts) Is M complete? Why or why not?
(d) (2 pts) Is M positive definite? Why or why not?
(e) (2 pts) Find an orthonormal eigenvector basis of \( \mathbb{R}^3 \) determined by M.
(f) (4 pts) M has a spectral factorization \( M = QAQ^{-1} \), write down \( Q \) and \( \Lambda \).

Solution:
(a) The characteristic equation here is \( (2 - \lambda)(\lambda^2 - 4\lambda + 3) = 0 \) and so \( \lambda_{1,2,3} = 1, 2, 3 \).

(b) Here we have \( \mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \), \( \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and \( \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \).

(c) Yes, since each eigenvalue is distinct and has exactly one eigenvector associated to it.

(d) Yes, since all the eigenvalues are positive definite.

(e) Note that since each eigenvalue is distinct, the corresponding eigenvectors are linearly independent and we have \( \mathbf{w}_1 = \begin{pmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \), \( \mathbf{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and \( \mathbf{w}_3 = \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \).

(f) We have \( Q = \begin{pmatrix} 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} \) and \( \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \).

Problem 5. (20 points) Consider the linear system

\[
\begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \end{bmatrix}
\]

(a) (5 pts) What is the compatibility condition of the matrix \( A \)?
(b) (10 pts) Find the general solution of the augmented system given above.
(c) (5 pts) Find the solution of minimum Euclidean norm of the system given above.

Solution:
(a) The compatibility condition for any vector \( \mathbf{b} \) is \( \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0 \)

(b) The general solution is \( \mathbf{x} = \begin{pmatrix} 2t - 3 \\ t \end{pmatrix} \) for any \( t \in \mathbb{R} \).

(c) Note that \( \ker(A) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) and we need \( \begin{pmatrix} 2t - 3 \\ t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0 \) which implies \( t = 6/5 \) so the solution of minimum Euclidean norm is \( \mathbf{x}^* = \begin{pmatrix} -3/5 \\ 6/5 \end{pmatrix} \).

Problem 6. (20 points) Write “TRUE” or “FALSE”. You do NOT need to justify your answer. Each part is worth 5 points.

(a) A linear transformation \( L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) maps circles to circles.
(b) If \( \|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\| \) then \( \mathbf{v}, \mathbf{w} \) are parallel vectors.
(c) Let \( K \) be the Gram matrix associated to vectors \( v_1, v_2, \ldots, v_n \in V \) where \( V \) is an inner product space, then \( K \) is positive definite if and only if \( v_1, v_2, \ldots, v_n \) span \( V \).
(d) The set \( W = \text{span}\{x^2 + 1, x^2 - 1, x^2 + x + 1, x^2\} \) is a vector space of dimension 4.
(e) The set of all $n \times n$ matrices $A$ such that $A^T = -A$ is a subspace of the vector space of all
$n \times n$ matrices, $\mathcal{M}_{n \times n}$.

Solution:
(a) False  (b) True  (c) False  (d) False  (e) True

Problem 7. (20 points) Show that if $Q$ is an orthogonal matrix, then $\|Qx\| = \|x\|$ for any vector $x \in \mathbb{R}^n$, where $\| \cdot \|$ denotes the standard Euclidean norm.

Solution:
$$\|Qx\| = \sqrt{(Qx,Qx)} = \sqrt{(Qx)^TQx} = \sqrt{x^TQ^TQx} = \sqrt{x^TIx} = \sqrt{x^Tx} = \|x\|$$

Problem 8. (20 points) Suppose that $A$ is any $n \times n$ nonzero matrix such that $A^T = -A$.

(a) (5 pts) Show that the diagonal entries of $A$ are zero.
(b) (5 pts) Show that $x^TAx = 0$ for any vector $x$.
(c) (10 pts) Is the matrix $B = I - A$ invertible? Why or why not?

Solution:
(a) Note that $A = -A^T$ implies $a_{ii} = -a_{ii}$ so $2a_{ii} = 0$ which implies $a_{ii} = 0$.

(b) Note that $x^TAx$ is a scalar, so it is equal to its own transpose, i.e.
$$x^TAx = (x^TAx)^T = x^TA^T x = -x^TAx$$
and so $x^TAx = -x^TAx$ and the only scalar with this property is 0, and thus $x^TAx = 0$.

(c) Yes, we show the rank($B$) = $n$ by showing ker($B$) = ker($I - A$) = $0$. Consider $(I - A)x = 0$ then this implies $x - Ax = 0$ and multiplying on the left by $x^T$ yields $x^Tx - x^TAx = 0$ and so $x^TAx = x^TAx$, and from part (b) we have $x^TAx = 0$, and so $x^Tx = 0$. But note that $\|x\|^2 = x^Tx$ and so we see that $\|x\|^2 = 0$ which implies $x = 0$. So we have that ker($B$) = $0$ and the Fundamental Theorem of Linear Algebra implies rank($B$) = $n$ so $B$ is invertible.