Problem #1 (16 points): Check the Caucy–Riemann (C-R) conditions for the following. If they satisfy the C-R conditions, find \( f'(z) \).

(a) \( f(x, y) = x - iy + 1 \).
(b) \( f(x, y) = y^3 - 3x^2y + iy(x^3 - 3xy^2 + 2) \).
(c) \( f(x, y) = e^y (\cos x + i \sin y) \).
(d) \( f(r, \theta) = \log r + i\theta \).

Problem #2 (6 points): Choose the constant \( a \) so that the function
\[
u(x, y) = x^3 + axy^2
\]
is harmonic. Find a conjugate function \( v \) in that case.

Problem #3 (16 points): Given the real part, \( u(x, y) \), of the an analytic function, \( f(z) = u(x, y) + i v(x, y) \), find the imaginary part, \( v(x, y) \).

(a) \( u(x, y) = 3x^2y - y^3 \).
(b) \( u(x, y) = 2x(c - y) \), where \( c \) is a constant.
(c) \( u(x, y) = \frac{y}{x^2 + y^2} \).
(d) \( u(x, y) = \cos x \cosh y \).

Problem #4 (24 points): Determine where, if anywhere, the following functions are analytic. Discuss if they have any singular points or if they’re entire.

(a) \( \tan z \).
(b) \( \exp(\sin z) \).
(c) \( \exp(1/(z - 1)) \).
(d) \( \exp(\bar{z}) \).
(e) \( \frac{z}{z^2 + 1} \).
(f) \( \cos x \cosh y - i \sin x \sinh y \).

Problem #5 (18 points): Consider the following complex potential
\[
\Omega(z) = -\frac{k}{2\pi z}, \quad k \in \mathbb{R},
\]
which is referred to as a doublet. Calculate the corresponding velocity potential, stream function, and velocity field. Sketch or use a computer to graph the streamlines.

Problem #6 (20 points): (Flow at a corner) Consider the complex potential
\[
\Omega(z) = z^c,
\]
where \( c > 1/2 \) is a constant. Writing \( z = re^{i\theta} \), show that the rays \( \theta = 0 \) and \( \theta = \pi/c \) are streamlines, and so can be thought of barriers. Calculate the corresponding velocity potential, stream function, and velocity field. Sketch or use a computer to graph the streamlines in the sector \( 0 \leq \theta \leq \pi/c \) for \( c = 4 \), \( c = 1 \), and \( c = 2/3 \).

Extra-Credit Problem #7 (4 points): Discuss the flow represented by \( \Omega(z) = \log z \). Setting \( z = re^{i\theta} \) gives
\[
\Omega(z) = \log r + i\theta + 2\pi ik, \quad k \in \mathbb{Z}
\]
(we’ll discuss the properties of \( \log z, z \in \mathbb{C}, \) next week). Calculate the corresponding velocity potential, stream function, and velocity field. Sketch or use a computer to graph the stream function. Discuss the results.

Extra-Credit Problem #8 (6 points): (Flow at a wall) Consider the complex potential
\[
\Omega(z) = c\log(z - a) + c\log(z + a) = c\log(z^2 - a^2) + 2k\pi ic, \quad k \in \mathbb{Z},
\]
where \( a > 0 \) and \( c \) real. Calculate the corresponding velocity potential, stream function, and velocity field. Sketch or use a computer to graph the stream function. Discuss the results.