Problem #1 (20 points): Find the branch points of the following functions and give a branch cut that will make each function single-valued:

(a) \((z - 1)^{-1/2}\)
(b) \((z + 1 - 2i)^{1/4}\)
(c) \(2 \log z^2\)
(d) \(z \sqrt{z}\)
(e) \(z^{1/3}(1 - z)^{2/3}\)

Problem #2 (10 points): Let \(\alpha\) be a real number. Show that the set of all values of the multivalued function \(\log(z^\alpha)\) is not necessarily the same as that of \(\alpha \log(z)\).

Problem #3 (16 points):

(a) Deduce the identity
\[\tanh^{-1} z = \frac{1}{2} \log \left(\frac{1 + z}{1 - z}\right).\]
(b) Use this identity to compute \(\frac{d}{dz} \tanh^{-1} z\).

Problem #4 (24 points):

(a) Show that the solution to Laplace’s equation
\[\nabla^2 T = \frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0\]
in the region \(-\infty < u < \infty\) and \(v > 0\), with the boundary conditions
\[T(u, v = 0) = \begin{cases} T_0, & u > 0 \\ -T_0, & u < 0 \end{cases}\]
is
\[T(u, v) = T_0 \left(1 - \frac{2}{\pi} \tan^{-1} \frac{v}{u}\right).\]
(b) Now we’ll use this result to solve Laplace’s equation in \(|z| < 1\) with the boundary conditions
\[T(r = 1, \theta) = \begin{cases} T_0, & 0 < \theta < \pi \\ -T_0, & \pi < \theta < 2\pi \end{cases}\]
Show that
\[w = i \left(\frac{1 - z}{1 + z}\right) \quad z = \frac{i - w}{i + w}\]
maps
- \(|z| \leq 1\) to the upper-half \(w\)-plane
- \(w = u + i v\) and \(v \geq 0\),
- \(r = 1, 0 < \theta < \pi\) onto \(v = 0, u < 0\), and
- \(r = 1, \pi < \theta < 2\pi\) onto \(v = 0, u > 0\).
(c) Use this mapping function to show that the solution of the boundary value problem in the circle is given by
\[T(x, y) = T_0 \left[1 - \frac{2}{\pi} \tan^{-1} \left(\frac{1 - (x^2 + y^2)}{2y}\right)\right]\]
or, in polar coordinates,
\[T(r, \theta) = T_0 \left[1 - \frac{2}{\pi} \tan^{-1} \left(\frac{1 - r^2}{2r \sin \theta}\right)\right].\]

Problem #5 (10 points): Let \(w(z) = (z^2 + 1)^{1/2}\), \((z - i) = r_1 e^{i\theta_1}\), and \((z + i) = r_2 e^{i\theta_2}\). Find which ranges for \(\theta_1\) and \(\theta_2\) would make the following branch cuts:

(a) \(\{ix : \kappa \in \mathbb{R} \text{ and } |\kappa| < 1\}\)
(b) \(\{ix : \kappa \in \mathbb{R} \text{ and } |\kappa| > 1\}\).

Problem #6 (10 points): (Constructing harmonic functions) Let \(f(z) = u(x, y) + iv(x, y)\) be analytic in the open region \(R\). Assume that \(u^2 + v^2 \neq 0\) in \(R\). Show that
\[\frac{uu_x + vv_x}{u^2 + v^2}\]
is harmonic in \(R\). [Hint: If \(w(z)\) is analytic, is \(w'(z)/w(z)\) also analytic?]

Problem #7 (10 points): Discuss the branch-point and branch-cut structure of
\[\log[z - (z^2 + 1)^{1/2}]\]
and state where it’s analytic.

Extra-Credit Problem #8 (4 points): If \(f\) is a entire function such that \(f(0) = f'(0) = 0\) and \(\text{Im } f'(z) = 6xy - 2x\), then find \(f(1)\).

Extra-Credit Problem #9 (6 points): Consider the multivalued function
\[f(z) = \left(\frac{8}{7} z^3 - \frac{64}{7}\right)^{1/3}\]
Show that \(z_1 = 2, z_2 = 2e^{2\pi i/3}\), and \(z_3 = 2e^{-2\pi i/3}\) are branch points. Is \(z = \infty\) a branch point? Explain.

Now consider the following branch cuts:

(a) A straight line between \(z_1\) and \(z_2\).
(b) A straight line between \(z_1\) and \(z_3\).

Let \(F(z)\) denote the branch where \(\text{Im } f(10) = 0\). What is the value of \(F(1)\)?

What’s the value of \(F(1)\) if we change \(b\) to being between \(z_2\) and \(z_3\)?