Problem #1 (16 points): Use the definition of complex integration to evaluate $\oint_C f(z) \, dz$, where $C = \{z(t) = e^{it} : 0 \leq t \leq 2\pi\}$ is the unit circle centered at the origin, for the following:

(a) $f(z) = z^2$
(b) $f(z) = \overline{z}^2$
(c) $f(z) = (z + 1)z^{-2}$
(d) $f(z) = (z - 1/2)^{-2}$
(e) $f(z) = e^z$

Problem #2 (12 points): Show that the integral $\oint_C z^{-2} \, dz$, where $C$ is a path beginning at $z = -a$ and ending at $z = b$, $a, b > 0$, is independent of path so long as $C$ doesn’t go through the origin. Explain why the real-valued integral $\int_{-a}^{b} x^{-2} \, dx$ doesn’t exist, but the value obtained by formal substitution of limits agrees with the complex integral above.

Problem #3 (12 points): Let $C = \{Re^{it} : 0 \leq t \leq \pi, R \in \mathbb{R}\}$ be an open upper semicircle of radius $R$ with its center at the origin. Consider $f(z) = (z^2 + a^2)^{-1}$ and $\int_C f(z) \, dz = \int_C \frac{dz}{z^2 + a^2}$, where $a \in \mathbb{R}$ and $R > a > 0$. Show that $|f(z)| \leq \frac{1}{R^2 - a^2}$ implies $\left| \int_C f(z) \, dz \right| \leq \frac{\pi R}{R^2 - a^2}$. Find the limit of $\int_C f(z) \, dz$ as $R \to \infty$.

Problem #4 (12 points): Let $I_R = \int_C \frac{e^{iz}}{z^2 + 1} \, dz$, where $C_R = \{Re^{it} : 0 \leq t \leq \pi\}$ is the semicircle with radius $R$ in the upper-half plane. Show that $\lim_{R \to \infty} I_R = 0$.

Problem #5 (12 points): Evaluate $\oint_C f(z) \, dz$ for $C = \{z(t) = e^{it} : 0 \leq t \leq 2\pi\}$ (the unit circle centered at the origin) for the following:

(a) $f(z) = e^{iz}$
(b) $f(z) = e^z$
(c) $f(z) = (z^2 - 4)^{-1}$
(d) $f(z) = \sqrt{z - 4}$

Problem #6 (12 points): Evaluate $\oint_C f(z) \, dz$ using partial fractions for $C = \{z(t) = e^{it} : 0 \leq t \leq 2\pi\}$ (the unit circle centered at the origin) for the following:

(a) $f(z) = \frac{1}{z(z^2 - 2)}$
(b) $f(z) = \frac{z}{z^2 - 1/9}$
(c) $f(z) = \frac{1}{z(z^2 + 1)(z - 2)}$

Problem #7 (12 points): We wish to evaluate the integral $I = \int_{0}^{\infty} e^{ix^2} \, dx$. Consider the contour $I_R = \int_C e^{iz^2} \, dz$, where $C = C_1 + C_R + C_2$, $C_1 = \{t : 0 \to R\}$, $C_R = \{Re^{it} \mid t : 0 \to \pi/4\}$, and $C_2 = \{te^{i\pi/4} \mid t : R \to 0\}$. Show that $I_R = 0$ and that $\lim_{R \to \infty} \int_{C_R} e^{iz^2} \, dz = 0$.

(Hint: Use $\sin x \geq 2x/\pi$ on $0 \leq x \leq \pi/2$.) Then deduce $\lim_{R \to \infty} \left( \int_{C_1} e^{iz^2} \, dz - e^{i\pi/4} \int_{0}^{R} e^{-t^2} \, dt \right) = 0$, and use the well-known result of real integration $\int_{0}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}/2$ to show that $I = e^{i\pi/4} \sqrt{\pi}/2$.

Problem #8 (12 points): Consider the integral $I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$. Evaluate this integral by considering $\int_{C_R} \frac{dz}{z^2 + 1}$, where $C = C_1 + C_R$, $C_1 = (-R, R)$, and $C_R = \{Re^{it} \mid t : 0 \to \pi\}$; that is, $C$ is the closed semicircle in the upper-half $z$-plane with endpoints at $z = -R$ and $z = R$ plus the x-axis. Hint: Use that $\left| \frac{1}{(z^2 + 1)} \right| = \frac{1}{2i} \left( \frac{1}{z + i} - \frac{1}{z - i} \right)$, and show that the integral $\int_{C_R} \to 0$ as $R \to \infty$. Verify your answer by usual integration in real variables.

Extra-Credit Problem #9 (6 points): Ablowitz & Fokas: 2.6.2 (a) and (d)

Extra-Credit Problem #10 (4 points): Ablowitz & Fokas: 2.6.3