Problem #1 (10 points): Use the Taylor series for 
\((1 + z)^{-1}\) about \(z = 0\) to find the Taylor series of 
\(\log(1 + z)\) about \(z = 0\) for \(|z| < 1\).

Solution: The Taylor series for \((1 + z)^{-1}\) is just the 
geometric series

\[
\frac{1}{1 + z} = \sum_{n=0}^{\infty} (-1)^n z^n,
\]
and we know that it converges uniformly in \(|z| < 1\).

Since \(\int (1 + z)^{-1} \, dz = \log(1 + z) + c\) and since the above 
series converges uniformly so we can integrate it 
term-wise, we find that

\[
\log(1 + z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}.
\]

Moreover, this series will converge uniformly for 
\(|z| < 1\).

Problem #2 (12 points): Use the Taylor series for 
\((1 - z)^{-1}\) about \(z = 0\) (which converges when \(|z| < 1\)) to 
find a series representation of \((1 - z)^{-1}\) that converges 
when \(|z| > 1\). Hint: \((1 - z)^{-1} = -[z(1 - 1/z)]^{-1}\).

Solution: We know that

\[
\frac{1}{1 - \eta} = \sum_{k=0}^{\infty} \eta^k,
\]
when \(|\eta| < 1\). Using the hint,

\[
\frac{1}{1 - z} = \frac{1}{z} \left( \frac{1}{1 - 1/z} \right) = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{z^k} = -\sum_{k=0}^{\infty} \frac{1}{z^{k+1}}
\]
covers when \(|1/z| < 1\) or \(|z| > 1\).

Problem #3 (12 points): Expand

\[ f(z) = \frac{1}{1 + z^2} \]

about \(z = 0\) in

(a) a Taylor series for \(|z| < 1\) and
(b) a Laurent series for \(|z| > 1\).

Solution:

(a) The Taylor series for \(|z| < 1\) is

\[
\frac{1}{1 + z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}.
\]

(b) The Laurent series for \(|z| > 1\) is

\[
\frac{1}{1 + z^2} = \frac{1}{z^2} \left( 1 + \frac{1}{1 + 1/z^2} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2(n+1)}}.
\]

Problem #4 (18 points): Expand

\[ f(z) = \frac{z}{(z - 2)(z + i)} \]
in a Laurent series about \(z = 0\) in the following regions:

(a) \(|z| < 1\)
(b) \(1 < |z| < 2\)
(c) \(|z| > 2\)

Solution: Using partial fractions, we see that

\[
\frac{z}{(z - 2)(z + i)} = \frac{4/5 - 2i/5}{z - 2} + \frac{1/5 + 2i/5}{z + i}.
\]

(a) For \(|z| < 1\),

\[
f(z) = \frac{-2/5 + i/5}{1 - z^2/2} + \frac{2/5 - i/5}{1 - iz} = \left( \frac{2 + i}{5} \right) \sum_{n=0}^{\infty} \frac{1}{2^n} \left( -i \right)^n z^n.
\]

(b) For \(1 < |z| < 2\),

\[
f(z) = \frac{-2/5 + i/5}{1 - z^2/2} + \frac{1/5 + 2i/5}{z} \left( \frac{1}{1 + iz} \right)
= \left( \frac{2 + i}{5} \right) \sum_{n=0}^{\infty} \frac{z^n}{2^n} \left( -i \right)^{n+1} \frac{1}{z^{n+1}}.
\]

(c) For \(|z| > 2\),

\[
f(z) = \frac{4/5 - 2i/5}{z} \left( \frac{1}{1 - 2/z} \right) + \frac{1/5 + 2i/5}{z} \left( \frac{1}{1 + iz} \right)
= \left( \frac{2 - i}{5} \right) \sum_{n=0}^{\infty} \frac{z^n}{2^n} \left( 2^{n+1} - (-i)^{n+1} \right) \frac{1}{z^{n+1}}.
\]

Problem #5 (30 points): Evaluate the integral 
\(\int_{C} f(z) \, dz\) where \(C\) is the unit circle centered at the origin for the following \(f(z)\):

(a) \(\frac{e^z}{z^3}\)
Solution: Here we look any singular points inside the unit circle. Then we find the function's Laurent series expansion about these points. The counter integral is then just $2\pi i c_{-1}$, where $c_{-1}$ is the residue.

(a) The only singular point of $z^{-3} e^z$ in the unit circle is the origin. Since
\[
\frac{e^z}{z^3} = \frac{1}{z^3} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=-3}^{\infty} \frac{z^n}{(n+3)!}
\]
is the Laurent series expansion, we see that $c_{-1} = 1/2$. Thus $\oint_C f(z) \, dz = \pi i$.

(b) The only singular point of $(z^2 \sin z)^{-1}$ in the unit circle $z = 0$. The Laurent series expansion about $z = 0$ is
\[
\frac{1}{z^2 \sin z} = \frac{1}{z^3} + \frac{7z}{6z} + \frac{7z}{360} + \cdots,
\]
so $c_{-1} = 1/6$ and $\oint_C f(z) \, dz = i\pi/3$.

(c) Since $\tanh z$ is analytic in the unit circle, $\oint_C f(z) \, dz = 0$.

(d) Since $\frac{1}{\cos z}$ has two singular points in the unit circle, $z = \pm \pi/4$, we find the Laurent expansion about these points
\[
f(z) = \frac{\pm 1}{2(\pi \pm \pi/4)} \pm \frac{(z \pm \pi/4)^3}{3} \pm \frac{7(z \pm \pi/4)^3}{45} + \cdots.
\]
Therefore, $\oint_C f(z) \, dz = 2\pi i (1/2 - 1/2) = 0$.

(e) Since
\[
e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots,
\]
we know that $\oint_C f(z) \, dz = 2\pi i$.

Problem #6 (18 points): Let
\[
\exp\left\{\frac{t}{2}(z-1/z)\right\} = \sum_{n=0}^{\infty} J_n(t) z^n
\]
define $J_n(t)$. Using the definition of Laurent series and the properties of integration, show that
\[
J_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin \theta)} \, d\theta
\]
\[
= \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta - t\sin \theta) \, d\theta
\]
The functions $J_n(t)$ are called Bessel functions and they're well-known special functions in mathematics and physics.

Solution: Here we just use the definition of the Laurent series:
\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz.
\]
Let $C$ be the unit circle and $z = e^{i\theta}$:
\[
J_n(t) = \frac{1}{2\pi i} \oint_C \frac{\exp\left\{\frac{t}{2}(z - 1/z)\right\}}{(z - z_0)^{n+1}} \, dz
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{\frac{t}{2}(e^{i\theta} - e^{-i\theta})\right\} i e^{i\theta} \, d\theta
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{\frac{t}{2}(e^{i\theta} - e^{-i\theta}) - i\theta\right\} \, d\theta
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t \sin \theta - n\theta + i \sin(t \sin \theta - n\theta)) \, d\theta
\]
\[
= \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta - t\sin \theta) \, d\theta,
\]
where we used that $\sin(t \sin \theta - n\theta)$ is an odd function while $\cos(t \sin \theta - n\theta)$ is an even function.

Extra-Credit Problem #7 (10 points): Show that
\[
e^{u/z + v/z} = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k,
\]
where
\[
a_k = \frac{1}{2\pi} \int_{0}^{2\pi} e^{(u+v)\cos \theta} \cos((u-v) \sin \theta - k\theta) \, d\theta
\]
and
\[
b_k = \frac{1}{2\pi} \int_{0}^{2\pi} e^{(u+v)\cos \theta} \cos((u-v) \sin \theta - k\theta) \, d\theta
\]

Solution: We follow almost the same procedure as #6 and consider $C$ to be the unit circle with $z = e^{i\theta}$:
\[
c_n = \frac{1}{2\pi i} \oint_C \frac{e^{u/z + v/z}}{(z - z_0)^{n+1}} \, dz
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{ue^{i\theta} + ve^{-i\theta}\right\} \, i e^{i\theta} \, d\theta
\]
\[
\begin{align*}
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left( u e^{i\theta} + v e^{-i\theta} - i n\theta \right) d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[ u \cos \theta + i \sin \theta \right. \\
&\left. + v (\cos \theta - i \sin \theta) - i n\theta \right] d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[ (u + v) \cos \theta + i (u - v) \sin \theta - i n\theta \right] d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(u + v) \cos \theta} \left\{ \cos [(u - v) \sin \theta - n\theta] \\
&\quad + i \sin [(u - v) \sin \theta - n\theta] \right\} d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(u + v) \cos \theta} \cos [(u - v) \sin \theta - n\theta] d\theta,
\end{align*}
\]
where we used that \(\sin [(u - v) \sin \theta - n\theta]\) is an odd function. Then letting \(a_k = c_k\) and \(b_k = c_{-k}\) gives exactly
\[
\begin{align*}
a_k &= \frac{1}{2\pi} \int_{0}^{2\pi} e^{(u + v) \cos \theta} \cos [(u - v) \sin \theta - k\theta] d\theta \\
&= \frac{1}{2\pi} \int_{0}^{2\pi} e^{(u + v) \cos \theta} \cos [(v - u) \sin \theta - k\theta] d\theta.
\end{align*}
\]