Problem #1 (24 points): Find all the singular points (be sure to check $z_\infty$) for the following functions:

(a) \( \frac{e^{z^2} - 1}{z^2} \),
(b) \( \frac{e^{2z} - 1}{z^2} \),
(c) \( e^{\tan z} \),
(d) \( \frac{z^3}{z^2 + z + 1} \),
(e) \( \log(1 + z^{1/2}) \),
(f) \( \text{sech} \ z \).

Then classify them as either isolated or non-isolated and by type (such as removable, pole of order $N$ and strength $c_\infty$, essential, branch point, cluster point, or natural barrier).

Solution:

(a) The point $z = 0$ is a removable singularity because we need $f'(0) = 1$ in order for this function to be analytic for $|z| < \infty$. Also,

\[
\frac{e^{z^2} - 1}{z^2} = 1 + \frac{z^2}{2} + \frac{z^4}{2!} + \frac{z^6}{3!} + \ldots - 1 = 1 + \frac{z^2}{2!} + \frac{z^4}{3!} + \ldots \quad \text{and so } z = \infty \text{ is an essential singularity as shown in the book.}
\]

(b) Since

\[
\frac{e^{2z} - 1}{z^2} = \frac{2z}{z^2} + \frac{2^3}{3!} z + \frac{2^4}{4!} z^2 + \ldots \quad \text{and thus } z = 0 \text{ is a simple pole. Let } z = 1/t \text{ so that } \frac{e^{2z} - 1}{z^2} = t^2 (e^{2/t} - 1). \text{ As } t \to 0, e^{2/t} \text{ has an essential singularity. Therefore, the point } z = \infty \text{ is an essential singularity.}
\]

(c) As was shown in the book (pg. 146) we can represent the tangent function as

\[
\tan z = -\frac{1}{z - (\pi/2 + n\pi)} + \frac{1}{3} (z - (\pi/2 + n\pi) + \ldots
\]

for $n \in \mathbb{Z}$. Since we can expand $e^{\tan z}$ in a Taylor series about $\tan z$,

\[
e^{\tan z} = \sum_{n=0}^{\infty} \frac{(\tan z)^n}{n!}
\]

the points $z = \pi/2 + n\pi$ are essential singularities. Let $z = 1/t$ and consider the

(b) The roots of $z^2 + z + 1$ are $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$ and so these are simple poles. Let $z = 1/t$ Then

\[
\frac{z^3}{z^2 + z + 1} = \frac{1}{t(1 + t + t^2)}
\]

From this we see $t = 0$ is a simple pole and so $z = \infty$ is a simple pole.

(e) We have branch points at $z = 0, \infty$. Now, let $z = t + 1$ and then we have two branches of the square root function $z = \pm \sqrt{t + 1}$, now, recall the Taylor series for the square root function,

\[
\sqrt{1 + t} = 1 + t/2 + t^2/8 + \ldots \quad \text{and so for the positive branch we have } 1 + \sqrt{t + 1} = 2 + t/2 + t^2/8 + \ldots\n\]

On the negative branch we have

\[
1 - \sqrt{1 + t} = -t/2 - t^2/8 - \ldots \quad \text{And so } t = 0 \text{ is a branch point as well, that is, } z = 1.
\]

(f) We want to find the singularities of \( \text{sech} \ z = \frac{1}{\cosh z} \) or when $\cosh z = 0 = e^z + e^{-z}$, or when $e^{2z} = -1$ Letting $-1 = e^{i\pi + 2\pi n}$ for $n \in \mathbb{Z}$ we find $z = i(\frac{\pi}{2} + n\pi)$ are simple poles. Now let $z = 1/t$ and now we look at when $\cosh(1/t) = 0$ This occurs when $e^{2/t} = -1$ and we need $t = \frac{1}{i(\pi/2 + n\pi)}$. So, along the imaginary axis the singularities cluster around $t = 0$ and thus $t = 0$ is a cluster point, that is, $z = \infty$ is a cluster point.

Problem #2 (20 points): Evaluate the integral $\oint_C f(z) \, dz$, where $C$ is a unit circle centered at the origin, for the following functions:

(a) \( \frac{g(z)}{z - \omega} \), where $g(z)$ is entire,
(b) \( \frac{z}{z^2 - \omega^2} \),
(c) \( ze^{1/z^2} \),
(d) \( \cot z \),
(e) \( \frac{1}{8z^2 + 1} \).

Solution:

(a) For $|\omega| > 1$ the integral is 0 by Cauchy's theorem. Otherwise, since $g$ is entire it has a Taylor series
centered at \( \omega \), that is \( g(z) = \sum_{n=0}^{\infty} a_n (z - \omega)^n \) and we have
\[
\oint_C \frac{g(z)}{z - \omega} \, dz = \oint_C \frac{\sum_{n=0}^{\infty} a_n (z - \omega)^n}{z - \omega} \, dz
\]
\[
= \oint_C \frac{a_0}{z - \omega} \, dz
\]
\[
= 2\pi i a_0 = 2\pi i g(\omega)
\]

(b) If \( |\omega| > 1 \) the integral is 0 by Cauchy's Theorem. Otherwise, expand the function,
\[
\frac{z}{z^2 - \omega^2} = \frac{1}{2} \left( \frac{1}{z + \omega} + \frac{1}{z - \omega} \right)
\]
and it follows
\[
\oint_C \frac{1}{2} \left( \frac{1}{z + \omega} + \frac{1}{z - \omega} \right) \, dz = \frac{2\pi i}{2}(1 + 1)
\]
\[
= 2\pi i
\]

(c) First, find the Laurent series for \( ze^{1/z^2} \),
\[
ze^{1/z^2} = z \left( 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \right)
\]
\[
= z + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} \cdots
\]

It follows
\[
\oint_C ze^{1/z^2} \, dz = \oint_C z + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} \cdots \, dz
\]
\[
= \oint_C \frac{1}{z} \, dz
\]
\[
= 2\pi i
\]

(d) Here we note that the only singularity of \( \cot z \) in the unit circle is \( z = 0 \) so we can expand about \( z = 0 \) in the usual manner. that is
\[
\frac{\cos z}{\sin z} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{\frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}}
\]
\[
= \frac{1}{z} \left( 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \right) \left( 1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \cdots \right)
\]
\[
= \frac{1}{z} + O(1)
\]
and so the integral is
\[
\oint_C \cot z \, dz = \oint_C \frac{1}{z} \, dz = 2\pi i
\]

(e) The zeros of \( 8z^3 + 1 \) are \( z^3 = -\frac{1}{8} = \frac{1}{2} e^{-\pi i + 2\pi n i} \), for \( n = 0,1,2 \). The roots are \( z = \frac{1}{2} e^{\pi i/3} \), \( z = \frac{1}{2} e^{\pi i/3} \) and \( z = -\frac{1}{2} \) and they all lie within the unit circle. Then
\[
\frac{1}{8z^3 + 1} = \frac{1}{(z + 1/2)(z - 1/2 e^{-\pi i/3})(z - 1/2 e^{\pi i/3})}
\]

Using partial fractions you can show
\[
\frac{1}{(z + 1/2)(z - 1/2 e^{-\pi i/3})(z - 1/2 e^{\pi i/3})} = \cdots
\]
\[
\frac{A}{z - z_0} + \frac{B}{z - z_1} + \frac{C}{z - z_2}
\]

where
\[
A = -\frac{1}{4(1 + e^{\pi i/3})(1 + e^{-\pi i/3})}
\]
\[
B = \frac{1}{4(1 + e^{\pi i/3})(e^{-\pi i/3} - e^{\pi i/3})}
\]
\[
C = \frac{1}{4(1 - e^{-\pi i/3})(e^{\pi i/3} - e^{-\pi i/3})}
\]

Computing \( \oint_C f(z) \, dz \) yields
\[
\oint_C \frac{A}{z - z_0} + \frac{B}{z - z_1} + \frac{C}{z - z_2} \, dz = 2\pi i(A + B + C)
\]
\[
= 0
\]

**Problem #3 (16 points):** Determine if the following functions are meromorphic. And if they’re meromorphic, determine the order, strength, and location of all their poles.

(a) \( \frac{z}{z^4 + 2} \)

(b) \( \frac{z}{\sin^2 z} \)

(c) \( \frac{e^z - 1 - z}{z^4} \)

(d) \( \tan z \)

**Solution:**

(a) This is a rational function and is meromorphic. It has four simple poles, one for each of the roots \( z^4 = -2 \),
\[
z_0 = 2^{1/4} e^{\pi i/4} = 2^{-1/4} (1 + i)
\]
\[
z_1 = 2^{1/4} e^{3\pi i/4} = 2^{-1/4} (-1 + i)
\]
\[
z_2 = 2^{1/4} e^{5\pi i/4} = 2^{-1/4} (-1 - i)
\]
\[
z_3 = 2^{1/4} e^{7\pi i/4} = 2^{-1/4} (1 - i)
\]
The function $\tan z$ is meromorphic and it's poles and strengths are outlined in the book (pg. 146). The poles are simple with strength $-1$ at $z = \pi/2 + m\pi$ for $m \in \mathbb{Z}$.

**Problem #4 (8 points):** Consider

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{w\, dw}{(w^2 - 2)(w - z)},$$

where $C$ is the unit circle centered at the origin. Evaluate the integral for $|z| < 1$ and then for $|z| > 1$; is one the analytic continuation of the other? If $f(z)$ is meromorphic, then find the location, order, and strength of its poles.

**Solution:** First, consider the function for $|z| < 1$. Here there are singularities where $z = w$ inside the unit circle (in $w$). From CIF we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} \, dw$$

and so in this case $f(w) = \frac{w}{w^2 - 2}$, that is

$$f(z) = \frac{z}{z^2 - 2}.$$

Using a partial fraction decomposition we have

$$f(z) = \frac{1/2}{z - \sqrt{2}} + \frac{1/2}{z + \sqrt{2}},$$

and so there are simple poles at $z = \pm \sqrt{2}$ of strength $1/2$.

For $|z| > 1$ the integral is 0 because there are no singularities inside the unit circle (in $w$). They cannot be the analytic continuation of each other because there is no open set where they agree, simply the point $z = 0$.

**Problem #5 (4 points):** How would you redefine

$$f(z) = \begin{cases} 
  z^{-2}(1 - \cos z), & z \neq 0 \\
  1, & z = 0
\end{cases}$$

so that it's analytic for $|z| < \infty$. 

and

$$\frac{z}{z^4 + 2} = \frac{z}{(z - z_0)(z - z_1)(z - z_2)(z - z_3)}$$

The strength at $z_0$ is

$$\phi(z_0) = \frac{z_0}{(z_0 - z_1)(z_0 - z_2)(z_0 - z_3)} = -2^{-5/2} i$$

The strength at $z_1$ is

$$\phi(z_1) = \frac{z_1}{(z_1 - z_0)(z_1 - z_2)(z_1 - z_3)} = 2^{-5/2} i$$

The strength at $z_2$ is

$$\phi(z_2) = \frac{z_2}{(z_2 - z_0)(z_2 - z_1)(z_2 - z_3)} = -2^{-5/2} i$$

The strength at $z_3$ is

$$\phi(z_3) = \frac{z_3}{(z_3 - z_0)(z_3 - z_1)(z_3 - z_2)} = 2^{-5/2} i$$

where $\phi$ is understood to be different depending on which pole we are looking at.

(b) Recall $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$. Let $z = z_0 + z'$ for $z = m\pi + z'$. Using an addition formula we have

$$\sin(m\pi + z') = \sin(m\pi)\cos z' + \cos(m\pi)\sin z' = (-1)^m \sin z'$$

and $\sin^2 z = \sin^2 z'$ This allows us to say the following:

$$\frac{z}{\sin^2 z} = \frac{z'}{(z' - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots)}$$

and so factoring out a $z'$ of each of the sine expansions and using our usual trick for geometric series we have

$$\frac{z}{\sin^2 z} = \frac{z' + m\pi}{z'^2} \left(1 + \frac{z'^2}{2!} - \frac{z'^4}{4!} + \frac{z'^6}{6!} - \ldots\right)$$

and so there are poles of order 2 at $z = m\pi$ and $C_{-2} = m\pi$ for $m \neq 0$ and simple pole at $z = 0$ (i.e. $m = 0$).

(c) $f(z) = e^z - 1 - z = \frac{z^4}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots$ and so the Laurent series for $f(z)$ is

$$f(z) = \frac{1}{2!z^2} + \frac{1}{3!z} + \frac{1}{4!} + \frac{z}{5!} + \ldots$$

It follows $z = 0$ is a pole of order 2 with strength 1/2. The point $z = \infty$ is an essential singularity as well.
Solution: The function away from \( z = 0 \) is
\[
\frac{1 - \cos z}{z^2} = \frac{1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \right)}{z^2} = \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots
\]
and so the point \( z = 0 \) is a removable singularity. In order to be analytic we would need to define \( f(0) = 1/2 \).

**Problem #6 (20 points):** Let \( a \) be an isolated singular point of \( f \). Prove the following:

(a) If
\[
\lim_{z \to a} (z - a)f(z) = 0,
\]
then \( a \) is a removable singularity.

(b) If there exists an \( m \in \mathbb{N}, m > 0 \) such that
\[
\lim_{z \to a} (z - a)^m f(z) = c_{-m} \neq 0,
\]
then \( a \) is a pole of order \( m \) with strength \( c_{-m} \).

**Solution:**

(a) Since \( a \) is an isolated singular point of \( f \) it has a Laurent series expansion about \( a \),
\[
f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n
\]
and so
\[
(z-a)f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^{n+1}
\]
Taking the limit as \( z \to a \) of this we have
\[
0 = \lim_{z \to a} \sum_{n=-\infty}^{\infty} c_n(z-a)^{n+1} = \lim_{z \to a} \left( \cdots + c_{-2} \frac{1}{z-a} + c_{-1} + c_0(z-a)\cdots \right)
\]
In order for each term in the negative expansion to be 0 we need \( c_n = 0 \) for \( n < 0 \) and so \( a \) is a removable singularity.

(b) We use similar reasoning to part (a). Assume the same Laurent series for \( f(z) \), then
\[
(z-a)^m f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^{n+m}
\]
We have
\[
\lim_{z \to a} \sum_{n=-\infty}^{\infty} c_n(z-a)^{n+m} = \lim_{z \to a} \left( \cdots + c_{-(m+1)} \frac{1}{z-a} + c_{-m} + c_{-(m-1)}(z-a)\cdots \right)
\]
and so for this limit to exist it must be the case that \( c_n = 0 \) for \( n < m \) and assuming \( c_{-m} \neq 0 \) we have a pole of order \( m \).

**Problem #7 (8 points):** Show that the function
\[
f(z) = e^{1/z} + \frac{1}{(z+1)^2(z-2)}
\]
has isolated singularities at \( z = -1 \), 0, and 2.
Specifically, that \( z = 2 \) is a simple pole, \( z = -1 \) is a pole of order 2, and \( z = 0 \) is an essential singularity.

**Solution:** Using the previous problem we consider
\[
\lim_{z \to -2} (z-2)^2 f(z) = \lim_{z \to -2} ((z-2)^2 e^{1/z} + \frac{1}{(z+1)^2(z-2)}) = \frac{1}{9} \neq 0
\]
and so \( z = 2 \) is a pole of order 1. Also, since
\[
\lim_{z \to -1} (z+1)^2 f(z) = \lim_{z \to -1} ((z+1)^2 e^{1/z} + \frac{1}{z-2}) = -\frac{1}{3} \neq 0
\]
it follows \( z = -1 \) is a pole of order 2.
Since the function \( g(z) = \frac{1}{(z+1)^2(z-2)} \) is holomorphic in some ball centered at 0 it has a Taylor series there, \( g(z) = \sum_{n=0}^{\infty} c_n z^n \) and so in this ball,
\[
f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n + \sum_{n=0}^{\infty} c_n z^n
\]
and so \( z = 0 \) is an essential singularity.

**Extra-Credit Problem #8 (6 points):** Show that the series
\[
f(z) = \sum_{n=0}^{\infty} \frac{z+i}{z-i}
\]
converges uniformly in \( |z+i| < 1 \). Then show that
\[
g(z) = \frac{1+i z}{2}
\]
is its analytic continuation to the finite \( z \)-plane. Why doesn’t \( f(z) \) converge uniformly for all finite \( z \) if \( g(z) \) is analytic for all finite \( z \)?
\textbf{Solution:} Since $f(z)$ is a geometric series, it converges whenever 
\[
\frac{|z + i|}{|z - i|} < 1,
\]
in particular, since $z - i = z + i - 2i$, when 
\[
\frac{|z + i|}{|z - i|} \leq \frac{|z + i|}{|z + i| - 2|} < 1.
\]

The last inequality is true only when $|z + i| < 1$. The circle of convergence cannot be larger because at $z = 0$ ($0 + i = 1$) the series diverges. Summing geometric series for $|z + i| < 1$, one gets 
\[
f(z) = \frac{1}{1 - (z + i)/(z - i)} = \frac{z - i}{-2i} = \frac{iz + 1}{2} = g(z)
\]
for $\forall z : |z + i| < 1$. Therefore $g(z)$ is the analytic continuation of $f(z)$ to the finite $z$-plane.

The sum $f(z)$ is not a Taylor series, so it \textit{not} converging for all finite $z$ doesn't imply that $g(z)$ has a singular point at $z = 0$; since $g(z)$ is analytic for all finite $z$, a Taylor series of $g$ about any finite $z_0$ will converge for all finite $z$. 
