Problem #1 (20 points): Evaluate the integral

\[ I = \frac{1}{2\pi i} \oint_C f(z) \, dz, \]

where \( C \) is the unit circle centered at the origin, for the following \( f(z) \):

(a) \( f(z) = \frac{z + 1}{2z^3 - 3z^2 - 2z} \)

(b) \( f(z) = z^{-1} \cosh(z^{-1}) \)

(c) \( f(z) = \frac{e^{-\cos z}}{4z^2 + \pi^2} \)

(d) \( f(z) = \frac{\log(z + 2)}{2z + 1} \), principal branch

(e) \( f(z) = \frac{z + z^{-1}}{z(2z - (2z)^{-1})} \)

Solution:

(a) Using partial fractions, we see that

\[ f(z) = -\frac{1}{2z} + \frac{1}{5(z + 1/2)} + \frac{3}{10(z - 2)}. \]

Thus, \( I = \frac{2\pi i}{2\pi i} \left[ \text{Res}(f; 0) + \text{Res}(f; 1/2) \right] = \frac{1}{5} - \frac{1}{2} = -\frac{3}{10}. \)

(b) Since \( z = 0 \) is the only singular point of \( f \), we do a Laurent series expansion about \( z = 0 \):

\[ f(z) = \frac{1}{z} \frac{e^{1/z} + e^{-1/z}}{2} = \frac{1}{2z} \sum_{n=0}^{\infty} \frac{1 + (-)^n}{n!z^n} = \frac{1}{2z} \sum_{n=0}^{\infty} \frac{1 + (-)^n}{n!z^{n+1}} = \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{(2m)!}z^{2m+1} \]

Thus, \( \text{Res}(f; 0) = 1 \) and \( I = 1. \)

(c) The singular points of \( f \) are where \( 4z^2 + \pi^2 = 0 \) or \( z = \pm i\pi \), which are outside the unit circle. Thus, \( f \) is analytic in the unit circle and \( I = 0. \)

(d) This \( f \) has a branch point at \( z = -2 \) and a simple pole at \( z = -1/2 \). Since we're integrating around the unit circle, \( I = \text{Res}(f; -1/2) = \log(3/2)/2. \) (If we weren't on the principal branch, then \( I = \log(3/2)/2 + 2n\pi i, n \in \mathbb{Z} \))

(e) Using partial fractions,

\[ f(z) = -\frac{2}{z} + \frac{5}{4} \left( \frac{1}{z - 1/2} + \frac{1}{z + 1/2} \right) \]

So we have simple poles at \( z = 0, z = \pm 1/2 \) and

\[ I = \text{Res}(f; 0) + \text{Res}(f; 1/2) + \text{Res}(f; -1/2) = -2 + \frac{5}{4} + \frac{5}{4} = \frac{1}{2}. \]

Problem #2 (10 points): For complex \( a \) with \(|a| < 1\), show that

\[ I(a) = \int_0^{2\pi} \frac{d\theta}{1 - 2ac \cos \theta + a^2} = \frac{2\pi}{1 - a^2}. \]

Now find the result when \(|a| > 1\). [Hint: Is \( I(1/b) = b^2 I(b) \)?]

Solution: Let \( z = e^{i\theta} \), then \( d\theta = dz/(iz) \) and \( \cos \theta = (z^2 + 1)/(2z) \). Thus,

\[ I(a) = -i \oint_C \frac{dz}{z[1 + a^2 - 2a(z^2 + 1)/(2z)]} = -i \oint_C \left( \frac{1}{(1-a^2)(z-a)} + \frac{1}{(a^2-1)(z-1/a)} \right) dz, \]

where \( C \) is the unit circle. If \(|a| < 1\), then

\[ I(a) = (-i)(2\pi i)\text{Res}(f; z = a) = \frac{2\pi}{1 - a^2}; \]

if \(|a| > 1\), then

\[ I(a) = (-i)(2\pi i)\text{Res}(f; z = 1/a) = \frac{2\pi}{a^2 - 1}. \]

Problem #3 (18 points): Let \( C \) be the unit circle centered at the origin. Evaluate the integral

\[ I = \frac{1}{2\pi i} \oint_C f(z) \, dz, \]

for the following \( f(z) \) in two ways: (i) enclosing the singular points inside \( C \) and (ii) enclosing the singular points outside \( C \) (by including the point at infinity). Do you get the same result in both cases?

(a) \( f(z) = \frac{z^2 + 1}{z^2 - a^2}, \quad a^2 < 1 \)

(b) \( f(z) = \frac{z^2 + 1}{z^3} \)

(c) \( f(z) = z^2 e^{-1/z} \)
Solution:

(a) Note that
\[ f(z) = 1 + \frac{1 + a^2}{2a(z - a)} - \frac{1 + a^2}{2a(z + a)}, \]
which as simple poles at \( z = \pm a \).

(i) Since these poles are inside \( C \),
\[ I = \text{Res}(f; a) + \text{Res}(f; -a) = (1 - 1) \frac{1 + a^2}{2a} = 0. \]

(ii) Since \( f \) is analytic outside \( C \), \( I = 0 \).

Both results are the same, as expected since \( f \) is rational.

(b) Here we note that
\[ f(z) = \frac{1}{z^3}, \]
which has a simple pole (and a triple pole) at \( z = 0 \).

(i) Since the poles are inside \( C \),
\[ I = \text{Res}(f; 0) = 1. \]

(ii) Since \( f \to 0 \) as \( |z| \to \infty \),
\[ I = \text{Res}(f; \infty) = \lim_{|z| \to \infty} z f(z) = 1. \]

Again, both results are the same.

(c) Since
\[ f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! z^{n+2}} = z^2 - z + \frac{1}{2 \cdot 6z} + \frac{1}{24z^2} + \cdots, \]
we have an essential singular point \( z = 0 \).

(i) Since \( z = 0 \) is inside \( C \),
\[ I = \text{Res}(f; 0) = -\frac{1}{6}. \]

(ii) Here,
\[ I \text{Res}(f(z); z = \infty) = \text{Res}(t^{-2} f(t^{-1}); t = 0) \]
and
\[ \frac{f(t^{-1})}{t^2} = \frac{e^{-t}}{t^4} = \sum_{n=0}^{\infty} \frac{(-1)^n n^{n-4}}{n!}, \]
\[ = \frac{1}{t^4} - \frac{1}{t^3} + \frac{1}{2t^2} - \frac{1}{6t} + \frac{1}{24} + \cdots. \]
So
\[ I = -\frac{1}{6}, \]
and both methods give the same results.

Problem #4 (24 points): What type of singularity do the following functions have at \( z_\infty \)?

(a) \( z^m, \ m \in \mathbb{N}^+ \)
(b) \( z^{1/3} \)
(c) \( (z^2 + a^2)^{1/2}, \ a^2 > 0 \)
(d) \( \log(z^2 + a^2), \ a^2 > 0 \)
(e) \( e^z \)
(f) \( z^2 \sin^{-1} \)
(g) \( \sin^{-1} z \)
(h) \( \log(1 - e^{1/z}) \)

Solution:

(a) Pole of order \( m \) and strength 1.
(b) Algebraic branch point (with three Riemann sheets).
(c) Simple pole of strength 1.
(d) Logarithmic branch point.
(e) Essential singularity.
(f) Simple pole of order 1.
(g) Logarithmic branch point (with doubly infinite Riemann sheets).
(h) Logarithmic branch point.

Problem #5 (16 points): Let \( C_R \) be a circle of radius \( R \) centered at the origin, and let \( f \) be analytic outside contour \( C_R \).

(a) Show that
\[ \frac{1}{2\pi i} \oint_{C_R} f(z) \, dz = \frac{1}{2\pi i} \oint_{C_\rho} f(t^{-1}) \, \frac{dt}{t^2}, \]
where \( C_\rho \) is a circle of radius \( 1/R \) enclosing the origin. For \( R \to \infty \), conclude that the integral can be computed using \( \text{Res}[f(t^{-1})/t^2; 0] \).

(b) Suppose \( f(z) \) can be expressed as the Laurent series
\[ f(z) = \sum_{j=-\infty}^{-1} a_j z^j. \]
Show that the integral in (a) equals \( a_{-1} \).

Solution:

(a) Let \( z = 1/t \) so \( dz = -t^{-2} \, dt \). Denoting \( z = Re^{i\theta} \) implies that \( t = e^{-i\theta}/R \) for \( 0 \leq \theta \leq 2\pi \) and \( C_R \to -C_\rho \). Thus
\[ \frac{1}{2\pi i} \oint_{C_R} f(z) \, dz = \frac{1}{2\pi i} \oint_{C_\rho} f(t^{-1}) \, \frac{dt}{t^2}. \]
Using this and that \( z_\infty \) maps to \( t = 0 \), we get that \( \text{Res}(f(z); \infty) = \text{Res}(f(1/t)/t^2; 0) \).
(b) Let \( z = 1/t \), then
\[
\frac{f(1/t)}{t^2} = \frac{1}{t^2} \sum_{j=1}^{\infty} a_{-j} t^j \\
= \sum_{j=1}^{\infty} a_{-j-1} t^j = \frac{a_{-1}}{t} + a_{-2} + a_{-3} t + \cdots.
\]

Thus, since \( C_p \) is an arbitrarily small neighborhood about \( t = 0 \), the integral is
\[
\text{Res}(f(1/t)/t^2; 0) = a_{-1}.
\]

Problem #6 (12 points): Assume that \( f \) and \( g \) are analytic outside a circle \( C_R \) of radius \( R \) centered at the origin and
\[
\lim_{|z| \to \infty} f(z) = C_1 \quad \text{and} \quad \lim_{|z| \to \infty} zg(z) = C_2,
\]
where \( C_1 \) and \( C_2 \) are constants. Show that
\[
\frac{1}{2\pi i} \oint_{C_R} g(z)e^{f(z)} \, dz = C_2 e^{C_1}.
\]

Solution: Since \( f \) and \( g \) are analytic outside \( C_R \), we know that
\[
\frac{1}{2\pi i} \oint_{C_R} g(z)e^{f(z)} \, dz = \text{Res}\left( g(z)e^{f(z)}; \infty \right).
\]

Now we use the following deductions:

- Since \( f \to C_1 \) as \( |z| \to \infty \) and \( C_1 \) is finite, \( e^f \to e^{C_1} \) as \( |z| \to \infty \) and \( e^{C_1} \) is finite.
- Since \( zg(z) \to C_2 \) as \( |z| \to \infty \), \( g \to 0 \) as \( |z| \to \infty \) and \( ge^f \to 0 \) as \( |z| \to \infty \) because \( e^{C_1} \) is finite.
- From class (and the book), if \( h \to 0 \) as \( |z| \to \infty \), then \( \text{Res}(h; \infty) = \lim_{|z| \to \infty} zh(z) \). Thus,
\[
\text{Res}(g e^f; \infty) = \lim_{|z| \to \infty} zg(z)e^{f(z)} = C_2 e^{C_1},
\]
which is exactly what we wanted to show.

Extra-Credit Problem #7 (10 points): Plot
\[
I(a) = -i\pi \oint_{C_a} \frac{e^z}{z(z^2 + \pi^2)} \, dz
\]
for \(-\infty < a < \infty\) and where \( C_a \) is the rectangle with corners \(-1 + ia, -1 + i(a + 4), 1 + i(a + 4), \) and \(1 + ia\).

Solution:

We have seven regions:

- For \( a < -4 - \pi \), there are no singular points in \( C_a \) so \( I(a) = 0 \).
- For \(-4 - \pi < a < -4 \), only the singular point \( z = -i\pi \) is in \( C_a \) and
\[
I(a) = -i\pi(2\pi i) \left. \frac{e^z}{3z^2 + \pi^2} \right|_{z=-i\pi} = 1.
\]
- For \(-4 < a < -\pi \), we have both \( z = -i\pi \) and \( z = 0 \) in \( C_a \) and
\[
I(a) = \left. \frac{2\pi^2 e^z}{3z^2 + \pi^2} \right|_{z=-i\pi} + \left. \frac{2\pi^2 e^z}{3z^2 + \pi^2} \right|_{z=0} = 3.
\]
- For \(-\pi < a < -\pi - 4 \), we have only the singular point at \( z = 0 \) in \( C_a \) and
\[
I(a) = \left. \frac{2\pi^2 e^z}{3z^2 + \pi^2} \right|_{z=0} = 2.
\]
- For \( 4 - \pi < a < 0 \), we have both \( z = 0 \) and \( z = i\pi \) in \( C_a \) and
\[
I(a) = \left. \frac{2\pi^2 e^z}{3z^2 + \pi^2} \right|_{z=0} + \left. \frac{2\pi^2 e^z}{3z^2 + \pi^2} \right|_{z=i\pi} = 3.
\]
- For \( 0 < a < \pi \), we have only the singular point at \( z = i\pi \) in \( C_a \) and
\[
I(a) = \left. \frac{2\pi^2 e^z}{3z^2 + \pi^2} \right|_{z=i\pi} = 1.
\]
- For \( a > \pi \), there are no singular points in \( C_a \) so \( I(a) = 0 \).

Note that I’ve assumed that the contour goes in the positive direction (despite giving the points in the negative direction). You will get full credit if you assume that the contour went in the negative directions, which just changes \( I(a) \) above to \(-I(a)\).