1 Instructions

Labs may be done in groups of 3 or less. One report must be turned in for each group and must be in PDF format. Labs must include each student’s:

- Name
- Student number
- Section number

This lab is due on Friday, April 25, 2014 at 5pm. Each lab must be turned in through D2L on the course page. When you submit the lab please include each group members information (name, student number and section number) in the comments. This allows us to search for a students lab. **Late labs will not be accepted.** Once the labs are graded you need to specifically check that your grade was entered. If you are missing a grade please bring the issue up with your TA within a week of grading.

The report must be typed (including equations). Simply answering the lab questions will not earn you a good grade. Please take time to write your report as up to 20% of your grade may be based on organization, structure, style, grammar, and spelling.

2 Introduction

This lab demonstrates the modeling of some naturally occurring phenomena by first order systems. There are four models through which the student must negotiate: (1) the predator-prey
system, (2) the logistic predator-prey system, (3) the predator-prey system with harvesting and (4) the predator-prey system with migration.

3 Introduction of the Predator-Prey Model

A struggle for existence inevitably follows from the high rate at which all organic beings tend to increase. Every being, which during its natural lifetime produces several eggs or seeds, must suffer destruction during some period of its life, and during some season or occasional year; otherwise, on the principle of geometrical increase, its numbers would quickly become so inordinately great that no country could support the product. Hence, as more individuals are produced than can possibly survive, there must in every case be a struggle for existence, either one individual with another of the same species, or with the individuals of distinct species, or with the physical conditions of life. It is the doctrine of Malthus applied with manifold force to the whole animal and vegetable kingdoms; for in this case there can be no artificial increase of food, and no prudential restraint from marriage. Although some species may be now increasing, more or less rapidly, in numbers, all cannot do so, for the world would not hold them....

The amount of food for each species of course gives the extreme limit to which each can increase; but very frequently it is not the obtaining food, but the serving as prey to other animals, which determines the average numbers of a species.¹

Vito Volterra and Alfred Lotka reduced Darwin’s concepts for predatory prey interactions into a mathematical model. The model is actually a first order system, called the predator-prey (or Lotka-Volterra) system:

\[
\begin{align*}
    x' &= -ax + bxy \\
    y' &= cy - dxy
\end{align*}
\]

This model is the one of the simplest of predator and prey interaction models. \(x\) represents the predator population and \(y\) represents the prey population; these variables reside in the first quadrant, called the population quadrant. The parameters, called rate constants, \(a\), \(b\), \(c\) and \(d\) are all positive. Without the cross terms \((xy)\), we can see that the predator population decays exponentially, \(x' = -ax\), and that the prey population grows exponentially, \(y' = by\).

The cross terms, \(bxy\) and \(dxy\), represent the interaction of the two species. Notice that, clearly, the predator population is affected positively and that the prey population is affected negatively by interaction. In other words, food promotes the predator’s growth rate while serving as food diminishes the prey’s growth rate.

The predator-prey system is a result of the Balance Law:

\[ \text{Net rate of change of a population} = \text{Rate in} - \text{Rate out} \]

Supposing that migration into and out of the community is negligible, the rate of change is simply the difference between the birth and death rates. These rates must be proportional to the population size. In other words:

\[
x' = R_1 x, \quad x(0) = x_0 \\
y' = R_2 y, \quad y(0) = y_0
\]

where \( R_1 \) and \( R_2 \) are the coefficients of proportionality and a measure of the contribution of the average individual of a species to the overall growth rate of that species. If \( R_1 \) and \( R_2 \) are constant, this setting would represent the growth or decay, depending on the signs of \( R_1 \) and \( R_2 \), of each of the species. Various choices of \( R_1 \) and \( R_2 \) determine different types of models of interactions. In the case of the predator-prey system, \( R_1 = -a + by \) and \( R_2 = b - cx \).

Listed below, for comparison, are some other interacting species models.

- **Overcrowding**
  \[
x' = -(A - ax + by)x \\
y' = (B - cx - dy)y
\]

- **Cooperation**
  \[
x' = (A + by)x \\
y' = (B + cx)y
\]

- **Competition**
  \[
x' = (A - ax - by)x \\
y' = (B - cx - dy)y
\]

4 The Model

Let \( x \) be the population of foxes (in hundreds) and let \( y \) be the population of rabbits (in hundreds) on an island. Note that \( x \) and \( y \) are functions of \( t \), time in days. In this scenario, the foxes are the only predators of the rabbits and the rabbits are the only prey for the foxes. Systems of differential equations that govern the changes in the population of these two species are:

- **The Predator Prey Model**
  \[
x' = -ax + bxy \\
y' = cy - dxy
\]

represents the simplest of the predator-prey models as described above.
• **Predator Prey Model with Harvesting**

\[
\begin{align*}
x' &= -ax + bxy - H_1x \\
y' &= cy - dxy - H_2y
\end{align*}
\]

is another modification of (1). In this scenario, the two species are harvested (by trappers who want to sell the animals’ pelts) in such a way that the amount caught per unit of time is proportional to the population. In the language of the Introduction Section, \( R_1 = (-a + by - H_1) \) and \( R_2 = (c - dx - H_2) \). This model is called constant-effort harvesting.

• **Predator Prey Model with Migration**

\[
\begin{align*}
x' &= -ax + bxy \\
y' &= cy - dxy + K \sin(\omega t)
\end{align*}
\]

is yet another modification of (1). In this scenario, we add the assumption of periodic emigration and immigration of prey. Suppose that, by some means, the prey can leave the island but the predators chose to stay. We use a sine function as the forcing term. The amplitude of the migration term is given by \( K \) and the period is determined by \( \omega \).

5 **Questions and Issues to Address**

Your lab report should answer the following questions and address the following issues. All plots and calculations mentioned in these questions should be included in the report.

For the following questions, let \( a = 1.5, b = 1.1, c = 2.5 \) and \( d = 1.4 \), unless otherwise stated.

1. Classify system (1). Briefly discuss all terms in system (1). For example, what does the coefficient to the \( xy \) term in \( x' \) represent?

2. Find all of the equilibria for (1).

3. Plot the flow (phase portrait) of (1).

   • For this question, create two plots, one for \(-6 < x < 6, -6 < y < 6\) and one for \(0 < x < 6, 0 < y < 6\).
   
   • Use the first plot to inspect the equilibria (solution curves are not needed here).

   • Classify the equilibrium point \((0,0)\). What is occurring at the other equilibrium point?

   • Plot some solution curves over the second vector field plot.

   • Don’t forget to label your axes!

4. Let \( x(0) = 0.5 \) and \( y(0) = 1.0 \). Use an ODE integrator (ode45 in Matlab) to plot the curves \( x(t) \) and \( y(t) \), called component curves. Overlay these plots and label the curves appropriately (you may want to use different colors for each curve). Perform the following:
Compare the component curves to the solution curves in #3. How do these plots relate to one another?

Compare the component curves to one another (i.e. Compare the fox population to the rabbit population).

Comment on the phase of these populations. i.e. Are they in phase or out of phase with one another? If a phase shift occurs, why? If a phase shift doesn’t occur, why not? (You do not need to give a mathematical justification, you need only give a physical justification in terms of the populations.)

Approximate the period and the amplitude of each component curve.

5. (For this question, \(a, b, c\) and \(d\) are arbitrary parameters; do not substitute their given values) Volterra’s Law of Averages states: In system (1), the average predator and prey populations over the period of a cycle are, respectively, \(c/d\) and \(a/b\). In other words, the average population of each species over any of the cycles is a fixed constant! Prove this result for the rabbit population (the result for the fox population is similar) by doing the following steps.

- Solve for \(y\) in \(x' = -ax + bxy\).
- Compute the average value of \(y\) over a cycle of period \(T\) (interval is \([0, T]\)).
- Note that \(x(T) = x(0)\), then, as noted above, you should obtain that the average value is \(a/b\).

6. What do you notice about the result you just proved (say you proved it for both rabbits: \(a/b\) and foxes: \(c/d\)) and the equilibrium solutions for the system (1) (substitute the values for \(a, b, c\) and \(d\))? Explain why this result makes sense.

7. (For this question, \(a, b, c\) and \(d\) are arbitrary parameters; do not substitute their given values) If we shift the origin, \((0, 0)\), of the frame of system (1) to the point \((c/d, a/b)\), and linearize about \((0, 0)\), we obtain:

\[
\begin{align*}
u' &= \frac{bc}{d} v \\
v' &= -\frac{ad}{b} u
\end{align*}
\]

For the method on how this system is linearized, refer to Farlow pp. 433.

- Linearize this system at the point \((0, 0)\).
- Compute the eigenvalues for this linearized system.
- Given that \(a, b, c\) and \(d\) are positive constants, what can you conclude about the behavior of the general solution to the linearized system.
- Classify the equilibrium.
Now substitute the given values of $a, b, c$ and $d$ and give a numerical value for the eigenvalues. With what period do the predator and prey populations vary? (You approximated this value in #4.)

8. Now consider system (2) with $H_1 = H_2 = H$ and $x(0) = 1.667, y(0) = 1.111$. Use an ODE integrator (ode45 in Matlab) and overlay the following plots.

- Plot the phase portrait for (2) with $H = 0.5$.
- Plot the phase portrait for (2) with $H = 1.5$.
- Plot the phase portrait for (2) with $H = 3$.

9. Comment on each of the plots. What affect did each value of $H$ have on each of the populations? Which population was affected most? Justify your answer.

10. Suppose that system (2) models a predator-prey system and that you are the harvester. Further suppose that $x(0) = 1.667$ and $y(0) = 1.111$ and that it is required that $0.4 < x(t) < 1.8$ and $0.5 < y(t) < 2.5$. Find values for the positive coefficients $H_1$ and $H_2$ to maintain each species within the prescribed bounds. Justify your reasons for choosing these values and verify, graphically, that the values are valid.

11. Consider system (3), set $K = 0.2$ and $w = 5/(2\pi)$. What can you say about the long-term behavior of solutions? Interpret your observations in terms of the behavior of the populations. To answer this question, do the following:

- Plot the component curves of (3) starting with the initial condition $(1.667, 1.111)$.
- Make sure you follow solutions long enough to be confident you are seeing the "long-term" behavior.

12. Which of the models do you think is the most accurate? Explain why. You may combine models, if you do, call this new model (4). Given the model you chose, what are its shortcomings? Offer suggestions for improvement of the model.