Homework 2 Standard Solutions

Calculus III

Fall 2014

1. Let \((-1, 5, -2), (-3, 7, 2),\) and \((1, 3, 6)\) be 3 vertices of a cube. Determine the coordinates of the center of the cube.

**Solution 1.**

Let the points \(A, B,\) and \(C\) be defined as follows: \(A = (-1, 5, -2),\)
\(B = (-3, 7, 2),\) and \(C = (1, 3, 6).\) These points can be seen plotted below:

![Figure 1: A, B, and C plotted in 3D space.](image1)

![Figure 2: Three possible lines of the cube.](image2)

As can be observed in figure 2 above, there are three possible lines that can be drawn between vertices of a cube; they are labeled \(S\) for side (the shortest line segment), \(D\) for diagonal, and \(C\) for center (the longest line segment). Note that \(C\) passes directly through the midpoint of the cube. So, if the magnitude of the line segments connecting points \(A, B,\) and \(C\) can be found (and are all different amounts), we can figure out which one passes through the center.

Solving for the line segments and their magnitudes, we get:

\[
\mathbf{AB} = B - A = (-3 - (-1), 7 - 5, 2 - (-2)) = (-2, 2, 4) \quad \Rightarrow \quad |\mathbf{AB}| = \sqrt{(-2)^2 + 2^2 + 4^2} = \sqrt{24}
\]
AC = C−A = (1−1, 3−5, 6−2) = (2, −2, 8) ⇒ |AC| = \sqrt{2^2 + (-2)^2 + 8^2} = \sqrt{72}

BC = C−B = (1−3, 3−7, 6−2) = (4, −4, 4) ⇒ |BC| = \sqrt{4^2 + (-4)^2 + 4^2} = \sqrt{48}

Since AC is the longest, it must be the line that passes through the center of the cube. Therefore, the center of the cube is halfway between points A and C:

Center = \frac{A+C}{2} = \left(-\frac{1+1}{2}, \frac{5+3}{2}, -\frac{2+6}{2}\right) = (0, 4, 2)

2. Consider the four points: (−1, −4, 1), (−1, 1, 2), (1, −1, 0), (2, 1, −1).

(a) Draw the shape

(b) Find the total surface area of the object.

(c) Reflect the point (−1, −4, 1) across the plane defined by the points (1, −1, 0), (2, 1, −1), (−1, 1, 2).

(d) What is the surface area of the new shape? (Hint: you don’t need to recalculate surface areas.)

Solution 2. (a) Define \( P_1 = (-1, -4, 1), P_2 = (-1, 1, 2), P_3 = (1, -1, 0), P_4 = (2, 1, -1). \)

(b) The surface area can be found by adding the areas of each of the 4 faces:

\[
A = \frac{1}{2}(|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}| + |\overrightarrow{P_1P_3} \times \overrightarrow{P_1P_4}| + |\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_4}| + |\overrightarrow{P_2P_3} \times \overrightarrow{P_2P_4}|)
\]
\[ \vec{P_1P_2} \times \vec{P_1P_3} = \begin{vmatrix} i & j & k \\ 0 & 5 & 1 \\ 2 & 3 & -1 \end{vmatrix} = -8i - j - 13k = \sqrt{168} \]

\[ |\vec{P_1P_2} \times \vec{P_1P_3}| = 8 = | -i + j + k| = \sqrt{3} \]

\[ |\vec{P_2P_3} \times \vec{P_2P_4}| = \begin{vmatrix} i & j & k \\ 0 & 5 & 1 \\ 3 & 5 & -2 \end{vmatrix} = -15i + j - 20k = \sqrt{459} \]

\[ |\vec{P_2P_3} \times \vec{P_3P_4}| = \begin{vmatrix} i & j & k \\ 2 & -2 & -2 \\ 3 & 0 & -3 \end{vmatrix} = |6i + 6k| = \sqrt{72} \]

\[ A = \frac{1}{2}(\sqrt{168} + \sqrt{3} + \sqrt{459} + \sqrt{72}) \approx 22.3 \]

(c) From (b) we found that \( \vec{P_2P_3} \times \vec{P_2P_4} = \langle 6, 6, 0 \rangle \), so the plane has normal vector \( \langle 1, 0, 1 \rangle \). Using that \( P_3 \) is a point in the plane:

\[ 0 = \langle x - 1, y + 1, z \rangle \cdot \langle 1, 0, 1 \rangle = x - 1 + z. \]

So the equation of the plane is \( x + z = 1 \). The line that passes through \( P_1 \) and is parallel to the normal is

\[ \langle x(t), y(t), z(t) \rangle = \langle -1 + t, -4, 1 + t \rangle \]

The reflected point \( P_r \) must lie on this line on the opposite side of the plane and be the same distance from the plane as \( P_1 \):
Plugging the parametrization into the plane equation, we see that the line intersects the plane when
\[ x(t) + z(t) = -1 + t + 1 + t = 1 \Rightarrow t = \frac{1}{2}, \]
so the reflected point is at \( t = 1: \)

\[ P_r = \langle x(1), y(1), z(1) \rangle = \langle 0, -4, 2 \rangle. \]

3. One calculation that you will see later in this course is the mass flux, \( \dot{m} \) (with units of mass per unit time), of a fluid across a surface. (You might imagine the surface to be a screen so the fluid can pass right through the surface.) Here we will consider a simple case of a planar surface with area \( A \) and unit normal vector \( \hat{n} \). The fluid has mass density \( \rho \) with units of mass per unit volume, and velocity vector \( \mathbf{V} \).

Later, you will find that for this simple case the mass flux can be calculated by \( \dot{m} = \rho A \mathbf{V} \cdot \hat{n}. \)

(a) If the velocity vector \( \mathbf{V} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \) calculate the outward mass flux across each of the six surfaces of the finite object in the first octant bounded by \( x = 0, y = 0, z = 0, \) and \( x = 1, y = 2, z = 3. \)

(b) What is the total outward mass flux across all six bounding surfaces of the object?

(c) How should one interpret total outward mass flux values that are greater than zero? Equal to zero? Less than zero?

**Solution 3.** (a) The object forms a box with 6 sides; \( x = 1, y = 2, x = 0, y = 0, z = 3, z = 0. \) The area for each side in the order listed is as follows:

\[ A_1 = 2 \cdot 3 = 6 \]
\[ A_2 = 1 \cdot 3 = 3 \]
\[ A_3 = 2 \cdot 3 = 6 \]
\[ A_4 = 1 \cdot 3 = 3 \]
\[ A_5 = 1 \cdot 2 = 2 \]
\[ A_6 = 1 \cdot 2 = 2 \]

The normals for each side are as follows:

\[ \hat{n}_1 = \langle 1, 0, 0 \rangle \]
\[ \hat{n}_2 = \langle 0, 1, 0 \rangle \]
\[ \hat{n}_3 = \langle -1, 0, 0 \rangle \]
\[ \hat{n}_4 = \langle 0, -1, 0 \rangle \]
\[ \hat{n}_5 = \langle 0, 0, 1 \rangle \]
\[ \hat{n}_6 = \langle 0, 0, -1 \rangle \]

Using the given formula for \( \hat{m} \) we get the flux for each side:

\[ \hat{m}_1 = \rho(6) \langle 1, 2, 3 \rangle \cdot \langle 1, 0, 0 \rangle = 6\rho \]
\[ \hat{m}_2 = \rho(3) \langle 1, 2, 3 \rangle \cdot \langle 0, 1, 0 \rangle = 6\rho \]
\[ \hat{m}_1 = \rho(6) \langle 1, 2, 3 \rangle \cdot \langle -1, 0, 0 \rangle = -6\rho \]
\[ \hat{m}_1 = \rho(3) \langle 1, 2, 3 \rangle \cdot \langle 0, -1, 0 \rangle = -6\rho \]
\[ \hat{m}_1 = \rho(2) \langle 1, 2, 3 \rangle \cdot \langle 0, 0, 1 \rangle = 6\rho \]
\[ \hat{m}_1 = \rho(2) \langle 1, 2, 3 \rangle \cdot \langle 0, 0, -1 \rangle = -6\rho \]

(b) The total outward flux across all boundaries is \( \hat{m} = 6\rho + 6\rho - 6\rho - 6\rho + 6\rho - 6\rho = 0 \).

(c) Total outward mass flux values that are greater than 0 mean that there is a source inside the box so that fluid is flowing out of the box. Total outward mass flux values less than 0 indicate a sink inside the box, so fluid is flowing into the box. A total outward mass flux value of 0 indicates that there is neither a source nor sink inside the box and the amount of fluid coming into the box is exactly the same as the amount of fluid flowing out of the box.