1. Romberg's integration method is based on successive applications of the Trapezoidal rule. Given several trapezoidal results $T_0$, $T_1$, $T_2$, ... (with step size halved between successive approximations), we create a table

\[
\begin{array}{c|ccc}
T_0 & & & \\
T_1 & S_1 & & \\
T_2 & S_2 & C_2 & \\
T_3 & S_3 & C_3 & D_3 \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]

and read off improved results down the main diagonal.

a. Quote (no derivation necessary) how the entries are obtained in the columns to the right of the leftmost one (which contains the trapezoidal results).

b. Suppose the function we are integrating possesses some singularity so that the trapezoidal error expansion is not as usual, but

\[
\int f(x) \, dx = T_n = ah^{3/2} + \beta h^2 + \gamma h^{5/2} + \delta h^3 + ...
\]

Determine how we need to modify the rules to determine the entries in the different Romberg columns.

2. Suppose we know a function $y = f(x)$ and its first three derivatives at both $x = 0$ and $x = 1$, and that we want to evaluate $\int_0^1 f(x) \, dx$ as accurately as possible. The following are four plausible approaches. Compare their orders of accuracy, i.e. determine how high degree polynomials they are exact for. If two of the methods have the same order, compare the size of their leading error coefficients:
a. Consider the Taylor expansions around \( x = 0 \) and \( x = 1 \), integrate both over the interval \([0,1]\), and use the average of these two results,

b. Integrate separately Taylor expansions over \([0,1/2]\) and \([1/2,1]\), and add the results,

c. Form the (generalized) Hermite interpolation polynomial that uses function value and three derivatives at each end, and integrate this polynomial exactly,

d. Use the first three terms in Euler-MacLaurin's formula.

Note: The resulting formulas can be written as follows:

a. \( \int_0^1 f(x) \, dx \approx \frac{1}{2} [f(0) + f(1)] + \frac{1}{4} [f'(0) - f'(1)] + \frac{1}{12} [f''(0) + f''(1)] + \frac{1}{48} [f'''(0) - f'''(1)] \),

b. \( \int_0^1 f(x) \, dx \approx \frac{1}{2} [f(0) + f(1)] + \frac{1}{8} [f'(0) - f'(1)] + \frac{1}{36} [f''(0) + f''(1)] + \frac{1}{84} [f'''(0) - f'''(1)] \),

c. \( \int_0^1 f(x) \, dx \approx \frac{1}{2} [f(0) + f(1)] + \frac{2}{25} [f'(0) - f'(1)] + \frac{1}{54} [f''(0) + f''(1)] + \frac{1}{768} [f'''(0) - f'''(1)] \),

d. \( \int_0^1 f(x) \, dx \approx \frac{1}{2} [f(0) + f(1)] + \frac{1}{15} [f'(0) - f'(1)] - \frac{1}{720} [f'''(0) - f'''(1)] \).

The orders can be established by 'brute force' testing with \( f(x) = 1, x, x^2, x^3, \ldots \).

DO NOT solve the problem this way, but use instead, as far as you can, general arguments based on how the methods were designed.

3. To accelerate the convergence of a positive infinite series, we might for example use Euler-MacLaurin's formula

\[
\sum_N f(n) \approx \frac{1}{2} \int f(x) \, dx + \frac{1}{2} f(N) - \frac{1}{12} f'(N) + \frac{1}{24} f''(N) - \frac{1}{720} f'''(N) + \frac{1}{30240} f^{(5)}(N) + \ldots
\]

or Gregory's formula

\[
\sum_N f(n) \approx \frac{1}{2} \int f(x) \, dx + \frac{1}{2} f(N) - \frac{1}{12} \Delta f(N) + \frac{1}{24} \Delta^2 f(N) - \frac{19}{720} \Delta^3 f(N) + \frac{3}{160} \Delta^4 f(N) - \frac{863}{60480} \Delta^5 f(N) + \ldots
\]

In the Euler-MacLaurin case, we can pick up the coefficients from the Taylor expansion

\[
\left\{ \frac{1}{x} + \frac{1}{1 - e^x} \right\} = \frac{1}{2} - \frac{x}{12} + \frac{x^2}{720} - \frac{x^4}{30240} + \ldots
\]

Determine the function we need to use in order to similarly pick up the Gregory coefficients, i.e.

\[
\left\{ \text{????????????} \right\} = \frac{1}{2} - \frac{x}{12} + \frac{x^2}{24} - \frac{19 x^3}{720} + \frac{3x^4}{160} - \frac{863 x^5}{60480} + \ldots
\]

Hint: The easiest solution follows from using the relations between the \( E, \Delta \) and \( D \) operators.
4. The following are six plausible approximations to the function $e^x$ over the interval $[-1,1]$: 

(A) $e^x \approx \sum_{k=0}^{n} \frac{x^k}{k!}$,

(B) $e^x \approx 1/\sum_{k=0}^{n} \frac{(-x)^k}{k!}$

(C) $e^x \approx (1 + \frac{x}{n})^n$ 

(D) $e^x \approx \frac{c_0}{2} + \sum_{k=1}^{n} c_k T_k(x)$ where $c_k$ are the Chebyshev coefficients $c_k = \frac{2}{n} \int_{0}^{\pi} e^{\cos t} \cos kt \, dt$

(E) $e^x \approx \{\text{polynomial obtained by interpolation at the nodes } x_k = -1 + \frac{2k}{n}, k = 0, 1, ..., n\}$

(F) $e^x \approx \sum_{k=-n}^{n} d_k e^{i\pi kx}$ where $d_k = \frac{1}{2} \int_{-1}^{1} e^x e^{-i\pi kx} \, dx$

The six figures below show the errors (exact minus approximation) for the six cases over the interval $[-1,1]$ when using $n = 9$. However, the order of the figures (i)-(vi) is different from the order of the approximations (A)-(F):

(i) 
(ii) 
(iii) 
(iv) 
(v) 
(vi)

Determine which error picture corresponds to which approximation. Give fully convincing explanations for how you arrive at each of the answers.
5. The Gaussian quadrature formula
\[
\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \pi \sum_{k=0}^{n-1} f(\cos \left(\frac{(2k+1)\pi}{2n}\right)) \quad \text{is unusual in that all its weights are the same.}
\]
Show that it indeed is exact for all polynomials up through degree 2n − 1.

**Hint:** One way to proceed starts by changing variable \(x = \cos t\) and then consider the trapezoidal rule on the resulting integral
\[
\frac{1}{2} \int_0^{2\pi} f(\cos t) dt.
\]

6. Multiple choice - for each question, mark below by a cross either True or False (i.e. not always correct). You do not need to give any explanations for your answers to this problem.

<table>
<thead>
<tr>
<th></th>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>The 'natural' cubic spline enforces that the third derivative is zero at the two ends of the interval.</td>
<td>☐</td>
</tr>
<tr>
<td>b.</td>
<td>The quadratic B-spline is non-zero over three intervals.</td>
<td>☐</td>
</tr>
<tr>
<td>c.</td>
<td>If we are given the coefficients of two polynomials (p_1(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_N x^N) and (p_1(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_N x^N), it requires (O(N^2)) arithmetic operations to calculate all the 2N + 1 coefficients in their product (p_1(x) \cdot p_2(x)).</td>
<td>☐</td>
</tr>
<tr>
<td>d.</td>
<td>The bisection algorithm for finding a root of a scalar function (f(x)) converges roughly equally fast whatever the order is of an isolated root.</td>
<td>☐</td>
</tr>
<tr>
<td>e.</td>
<td>Newton's method works also for finding complex roots of a complex-valued function.</td>
<td>☐</td>
</tr>
<tr>
<td>f.</td>
<td>The best least squares approximation to a function (f(x)) by a constant (y = c) over an interval ([a, b]) is given by the relation (\int_a^b (f(x) - c) dx = 0).</td>
<td>☐</td>
</tr>
<tr>
<td>g.</td>
<td>Whenever a set of functions (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)) are linearly independent over an interval ([a, b]), there will be a unique solution to the problem of finding a best least square approximation to a function (f(x)) of the form (f(x) \approx c_1 \varphi_1(x) + c_2 \varphi_2(x) + \ldots + c_n \varphi_n(x)).</td>
<td>☐</td>
</tr>
<tr>
<td>h.</td>
<td>Same problem as in Part (g) just above, but consider the max norm instead of the least squares norm.</td>
<td>☐</td>
</tr>
<tr>
<td>i.</td>
<td>If we want to find the exact result for the finite sum (\sum_{k=0}^{N} k^{10}), Euler MacLaurin's formula offers a convenient option.</td>
<td>☐</td>
</tr>
<tr>
<td>j.</td>
<td>If an infinitely differentiable function (f(x)) satisfies both (f(0) = f'(0) = f''(0) = \ldots = 0) and (f(1) = f'(1) = f''(1) = \ldots = 0), then it holds that (f(x)) is everywhere zero over the interval ([0, 1]).</td>
<td>☐</td>
</tr>
<tr>
<td>k.</td>
<td>If data is tabulated at equispaced locations, Stirling's interpolation formula offers a very effective approach for inverse interpolation.</td>
<td>☐</td>
</tr>
</tbody>
</table>
l. If the rate of convergence for an iterative method is $O(2^{-n})$, the convergence is described as **quadratic**.

m. Considering the fixed point iteration $x_{n+1} = g(x_n)$, the condition $|g'(x)| < 1$ ensures that there cannot be two or more fixed points.

n. The Cauchy-Schwartz inequality provides a bound on the size of the scalar product of two functions.

o. The Chebyshev polynomials are orthogonal over $[0,1]$ with respect to the weight function $w(x) = 1/\sqrt{1-x^2}$.

p. If the polynomials $p_f(x)$ and $p_g(x)$ are the best least square approximations of some degree $n$ to the functions $f(x)$ and $g(x)$, respectively, then their sum $p_f(x) + p_g(x)$ is the best least square approximation of the same degree to the function $f(x) + g(x)$.

q. If we approximate an integral by Simpson's method, first using a step length $h$ and then $h/2$, we would (for $h$ small) expect the latter result to be about 8 times more accurate (function assumed to be non-periodic).

r. Pascal's triangle provides the necessary coefficients for 'economizing' a Taylor series.

s. Both the trapezoidal rule and Simpson's method are special cases of the Newton-Cotes family of quadrature methods.

t. An orthogonal polynomial $\varphi_n(x)$ can never have a double root.

u. The formula for the error in polynomial interpolation can be written $f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \psi(x)$ where $\psi(x) = \prod_{k=0}^{n}(x-x_k)$ and $\xi$ is some location on the interval that contains the node points $x_k$, $k = 0, 1, \ldots, n$.

v. Radial basis functions (RRBFs) using the radial function $\phi(r) = r^3$ produce in 1-D cubic splines.

w. Simpson's rule converges faster than the trapezoidal rule for approximating $\int_{-1}^{1} e^{\cos x} \, dx$.

x. The Hermite polynomials are orthogonal over $[0, \infty]$ with the weight function $w(x) = e^{-x}$.

y. A 5-node Gaussian quadrature formula would typically be exact for all polynomials up through degree 10.

z. When close to a simple root for a scalar function, a secant iteration usually reduces the error more than does a Newton iteration.