Write your name and your professor’s name on the front of your exam. You’re allowed one side of one sheet of letter-sized notes. You are not allowed to use textbooks, class notes, or a graphing calculator, but you may use a scientific calculator. To receive full credit on a problem you must show sufficient justification for your conclusion.

1. The following questions are independent from each other.

(a) Let $A$ be a $2 \times 2$ real matrix. Compute the solution to $Ax = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ if the following are true:

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ solves } Ax = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ solves } Ax = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ solves } Ax = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

**Solution:** Since $\begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ we have by the Superposition Principle

$$x = x_1 + 2x_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(b) Write down a basis for the space of real symmetric $2 \times 2$ matrices. What is the dimension of the space?

**Solution:** This is the space of all matrices of the form $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for $a, b, c \in \mathbb{R}$.

There are many choices for a basis for this space, but the simplest is

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then we can write any arbitrary symmetric $2 \times 2$ matrix as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Since the basis contains three elements the dimension of the space is 3.
2. The following questions are TRUE/FALSE. If your answer is TRUE you must briefly explain why it is true. If it is FALSE you must briefly explain why it is false, or give a counterexample.

(a) TRUE or **FALSE**: If matrix $A$ has all zeros on its main diagonal, then it is singular.

**Solution**: The statement is false. Consider the counterexample $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) TRUE or **FALSE**: For a square matrix $A$, if $x \in \ker A^2$ then $x \in \ker A$

**Solution**: The statement is false. Consider the case when $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Then, for instance, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \ker A^2$ but $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq 0$

(c) **TRUE** or FALSE: Let $A$ be an $n \times n$ matrix. If $A x_1 = A x_2$, with $x_1 \neq x_2$, then $A$ is not invertible.

**Solution**: The statement is true. To see this note that

$$A x_1 = A x_2 \iff A (x_1 - x_2) = 0$$

Since $(x_1 - x_2) \neq 0$ the vector is a nontrivial element of $\ker A$ so $A$ is not invertible.

(d) **TRUE** or FALSE: The following vectors span $\mathbb{R}^2$: $v_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$

**Solution**: The statement is true. Putting the vectors in a matrix and row reducing we have

$$\begin{bmatrix} 2 & -1 & -3 \\ 4 & -5 & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -3 \\ 0 & -3 & 0 \end{bmatrix}$$

The reduced matrix has two pivots so the vectors span $\mathbb{R}^2$. 
3. Consider the system \( Ax = b \) where \( A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & -1 \\ -2 & 0 & 0 \end{bmatrix} \) and \( b = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \).

(a) Find matrices \( L, U \) and \( P \) such that \( PA = LU \).

(b) Use the result of (a) to solve the linear system for \( x \).

(c) Now consider the slightly modified matrix \( B = \begin{bmatrix} 1 & 2 & -1 & 2 \\ -1 & -2 & -1 & 0 \\ -2 & 0 & 0 & -2 \end{bmatrix} \). Find bases for the following:

(i) \( \text{rng} \ B \)  
(ii) \( \text{corng} \ B \)  
(iii) \( \text{ker} \ B \)  
(iv) \( \text{coker} \ B \)

**Solution:**

(a) To find the LU decomposition of \( A \) we do Gaussian Elimination. We have

\[
\begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & -1 \\ -2 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1R_1 + R_2 \rightarrow R_2 \\ 2R_1 + R_3 \rightarrow R_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

So \( PA = LU \) \( \iff \)

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & -1 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & -2 \\ 0 & 0 & -2 \end{bmatrix}
\]

(b) We have \( Ax = b \) \( \Rightarrow \) \( PAx = Pb \) \( \Rightarrow \) \( LUx = Pb. \)

Applying our \( P \) to \( b \) we have \( Pb = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \). Then solving \( Ly = Pb \) we have

\[
\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 \\ 2 + 2(1) = 4 \\ -1 + 1(1) = 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}
\]

Then solving \( Ux = y \) we have

\[
\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & -2 \\ 0 & 0 & -2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 - 2(1) + 1(0) = -1 \\ 4 + 2(0) / 4 = 1 \\ 0 / 2 = 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = x
\]
(c) We notice that the matrix $B$ is exactly $A$ with an extra column tacked on. Thus, if we did Gaussian Elimination on $B$ we would get the same thing for the first three columns of the reduced system. We just need to apply the same row operations on the last column of $B$. If we do this we obtain

$$\begin{bmatrix}
1 & 2 & -1 & 2 \\
-1 & -2 & -1 & 0 \\
-2 & 0 & 0 & -2 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & -1 & 2 \\
0 & 4 & -2 & 2 \\
0 & 0 & -2 & 2 \\
\end{bmatrix}$$

(i) There are pivots in the first three columns of the reduced system. Then a basis for $\text{rng } B$ is made up of the first three columns of the original matrix $B$:

$$\text{rng } B = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

(ii) Since there are pivots in the first three rows of the reduced system, these rows make up a basis for $\text{corng } B$:

$$\text{corng } B = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right\}$$

(iii) To find a basis for $\text{ker } B$ we use the reduced system to write the basic variables in terms of the one free variable $x_4$:

$$x_3 = \frac{-2x_4}{-2} = x_4$$
$$x_2 = \frac{(2x_3 - 2x_4)/4}{0} = 0$$
$$x_1 = \frac{-2x_2 + x_3 - 2x_4}{-2x_2 + x_3 - 2x_4} = -x_4$$

So the kernel element is given by

$$x = x_4 \begin{bmatrix}
-1 \\
0 \\
1 \\
\end{bmatrix} \Rightarrow \text{ker } B = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(iv) From the Fundamental Theorem of Linear Algebra we know that $\dim \text{coker } = m - r$ where $r = \text{rank } A = \text{rank } A^T$. Since there are three pivots in the reduced version of $A$ we have $r = 3$. Then, since $A$ has $m = 3$ rows we have

$$\dim \text{coker } B = m - r = 3 - 3 = 0 \Rightarrow \text{coker } B = \text{span} \left\{ 0 \right\}$$
4. This problem is **NOT REQUIRED**. Do **NOT** even think of attempting this problem unless you have completed problems 1-3 to your complete satisfaction.

   (a) Show that if $A$ is $n \times n$, then $\det (-A) = (-1)^n \det A$

   (b) Prove that for odd $n$, any $n \times n$ skew-symmetric matrix (satisfying $A^T = -A$) is singular.

   (c) Construct a nonsingular skew-symmetric matrix.

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**Solution:**

(a) To change $-A$ to $A$ we have to multiply each row by $-1$. Each time we multiply a row by $-1$ the determinant is scaled by $-1$. Since we have $n$ rows this gives

$$\det (-A) = (-1)^n \det A$$

(b) We have

$$\det A = \det A^T = \det (-A) = (-1)^n \det A = -\det (A) \text{ since } n \text{ is odd}$$

From this we see that

$$\det A = -\det A \iff 2\det A = 0 \iff \det A = 0 \iff A \text{ is singular}$$

(c) We know that we need $n$ even for skew-symmetric $A$ to be nonsingular. The simplest example is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$