1. Prove that Newton's method for finding a simple root to \( f(x) = 0 \) is quadratically convergent (assuming that \( f(x) \) is sufficiently smooth).

Solution:

The Newton iteration is \( x_{n+1} = x_n - f(x_n) / f'(x_n) \).

**Approach 1:** Taylor expand around the root. The relations:
\[
x_n = p + \epsilon_n \\
f(x_n) = f(p + \epsilon_n) = f(p) + \epsilon_n f'(p) + O(\epsilon_n^2),
\]
\[
f'(x_n) = f'(p + \epsilon_n) = f'(p) + O(\epsilon_n)
\]
imply (noting that \( f'(p) = 0 \))
\[
\frac{f(x_n)}{f'(x_n)} = \frac{\epsilon_n f'(p) + O(\epsilon_n^2)}{f'(p) + O(\epsilon_n)} = \epsilon_n \frac{1 + O(\epsilon_n)}{1 + O(\epsilon_n)} = \epsilon_n (1 + O(\epsilon_n)), \text{ and therefore}
\]
\[
p + \epsilon_{n+1} = p + \epsilon_n - \epsilon_n (1 + O(\epsilon_n)) \Rightarrow \epsilon_{n+1} = O(\epsilon_n^2)).
\]

**Approach 2:** Consider fixed point theory: Write the Newton iteration as \( x_{n+1} = g(x_n) \) with \( g(x) = x - f(x) / f'(x) \). Then \( g'(x) = 1 - \frac{f'(x)^2 - f(x) f''(x)}{f'(x)^2} \). At the root, this evaluates to \( g'(p) = 1 - \frac{f'(p)^2}{f'(p)^2} = 0 \). Thus:
\[
g(x_n) = g(p + \epsilon_n) = g(p) + 0 \cdot \epsilon_n + O(\epsilon_n^2), \text{ and}
\]
\[
\epsilon_{n+1} = x_{n+1} - p = g(x_n) - p = O(\epsilon_n^2).
\]

2. At the three locations \( x = -h, 0, +h \), the function \( f(x) \) takes the values \( f(-h), f(0), f(h) \), respectively. Find the interpolating quadratic polynomial \( p_2(x) \) to these three data points and calculate its first derivative at \( x = 0 \). The result should take the form
\[
p_2'(0) = w_{-1} f(-h) + w_0 f(0) + w_1 f(h)
\]
where the weights \( w_{-1}, w_0, w_1 \) do not depend on the function \( f \).
Solution:
Either Lagrange’s or Newton’s formulas give us the interpolation polynomial. In the Lagrange case, we get
\[
p_z(x) = \frac{(x - 0)(x - h)}{(-h - 0)(-h - h)} f(-h) + \frac{(x + h)(x - h)}{(0 + h)(0 - h)} f(0) + \frac{(x + h)(x - 0)}{(h + h)(h - 0)} f(h)
\]
\[
= \frac{1}{2h^2} x(x - h) f(-h) - \frac{1}{h^2} (x^2 - h^2) f(0) + \frac{1}{2h^2} x(x + h) f(h).
\]
Trivial algebra then gives
\[
p_z'(0) = -\frac{1}{2h} f(-h) + \frac{1}{2h} f(h),
\]
i.e. the weights become $w_{-1} = -\frac{1}{2h}$, $w_0 = 0$, $w_i = \frac{1}{2h}$.

3. The figure below illustrates (on different scales) the cubic $B$-spline $B(x)$, and the cubic cardinal spline $C(x)$, listing their function values at the equispaced node points.

When these splines are normalized as shown, demonstrate that \( \int_{-\infty}^{\infty} B(x) \, dx = \int_{-\infty}^{\infty} C(x) \, dx = 1 \).

Hint: The problem can be solved quickly, with no knowledge needed about the individual cubics involved.

Solution:
The same argument applies to both cases, so let’s describe it for $B(x)$. Consider a sum of translates of $B(x)$: $s(x) = \sum_{k=-N}^{N} B(x - k)$. For $N$ large, this spline $s(x)$ will take the value 1 over a central interval of length $2N + O(1)$, so $\int_{-\infty}^{\infty} s(x) \, dx = 2N + O(1)$. It also has to equal 
\[
(2N + 1) \int_{-\infty}^{\infty} B(x) \, dx.
\]
For $N$ increasing, the coefficient for $N$ has to match in the two expressions, implying the result \( \int_{-\infty}^{\infty} B(x) \, dx = 1 \).
4. **Multiple choice** - for each question, mark below by a cross either True or False (i.e. not always correct). You do not need to give any explanations for your answers to this problem.

<table>
<thead>
<tr>
<th></th>
<th>True</th>
<th>False</th>
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<tbody>
<tr>
<td>a. If $x_i, i = 1, 2, 3, 4$ are distinct (no two the same), then $\det \begin{bmatrix} 1 &amp; x_1 &amp; x_1^2 &amp; x_1^3 \ 1 &amp; x_2 &amp; x_2^2 &amp; x_2^3 \ 1 &amp; x_3 &amp; x_3^2 &amp; x_3^3 \ 1 &amp; x_4 &amp; x_4^2 &amp; x_4^3 \end{bmatrix} \neq 0$</td>
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<td>b. If a large number of interpolation nodes are equispaced over an interval, interpolation with a single polynomial is likely to be very inaccurate near the end points. This is known as the Gibbs phenomenon.</td>
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<td>c. For $n$ large, $\sum_{k=1}^{n} k^3 = \frac{1}{4} n^4 + O(n^3)$</td>
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<td>d. The operators defined by $E f(x) = f(x + h)$ and $D f(x) = f'(x)$ are linked by the relation $E = e^D$.</td>
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<td>e. The linear $B$-spline is non-zero just at a single node point.</td>
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<td>f. When approximating a function with a natural cubic spline, the error is larger near the ends of the interval than near the center.</td>
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<td>g. By Gersgorin’s theorem, we can immediately conclude that the matrix $\begin{bmatrix} 3 &amp; 1 &amp; -1 \ 1 &amp; -8 &amp; 1 \ -1 &amp; 6 &amp; 3 \end{bmatrix}$ is non-singular.</td>
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<td>h. A cubic $B$-spline will never become negative, no matter how its (distinct) nodes are distributed.</td>
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<td>i. If the rate of convergence for an iterative method is $O(2^{-n})$, the convergence is described as quadratic.</td>
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<td>j. The Fourier transform is by convention defined over the interval $[0, \infty]$.</td>
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<tr>
<td>k. Consider the fixed point iteration $x_{n+1} = g(x_n)$. If it holds that $</td>
<td>g(x)</td>
<td>\leq 1$, then that there cannot be two or more fixed points.</td>
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<tr>
<td>l. If $u(x)$ is real valued, and we define its Fourier coefficients $\hat{u}<em>k$ by $u(x) = \sum</em>{k=-\infty}^{\infty} \hat{u}<em>k e^{ikx}$, then $\hat{u}</em>{-k}$ is the complex conjugate of $\hat{u}_k$.</td>
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<td>m. The $N^{th}$ root of unity is defined by $\omega = e^{\pi i/N}$.</td>
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n. Similarly to the cubic spline needing two ‘extra’ end conditions, one condition suffices for a quadratic spline.

o. If we use Matlab’s default spline routine, and provide it with five node locations and the correct values for the \( B \)-spline at these five nodes, the spline it creates will be the \( B \)-spline.

p. Raising the DFT matrix to the fourth power will produce a multiple of the identity matrix.