Homework 13
APPM 5440 Fall 2014 Applied Analysis

Due date: Friday, Nov. 21 2014, before 10 AM

Theme: Orthogonality in Hilbert Spaces, §6.2 and 6.3

Instructions: Problems marked with "Collaboration Allowed" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to post requests for help on forums such as http://math.stackexchange.com/. On these problems, please write down the names of the students that you worked with.

On problems marked “No Collaboration,” collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Problem 1: No Collaboration. Lemma 1.3 from the handout: Let $M$ be a non-empty subset of a Hilbert space $\mathcal{H}$. Then the span($M$) is dense in $\mathcal{H}$ if and only if $M^\perp = \{0\}$.

Problem 2: No Collaboration Set $I = [-1,1]$ and consider the Hilbert space $\mathcal{H} = L^2(I)$. Let $M$ denote the subspace of $\mathcal{H}$ consisting of all even functions (e.g., functions $g$ such that $g(x) = g(-x)$). Given a fixed $f \in \mathcal{H}$, prove that

$$\inf_{g \in M} \|f - g\| = \left( \int_{-1}^{1} \frac{|f(x) - f(-x)|^2}{2} \, dx \right)^{1/2}.$$

(Don’t worry about any possible issues relating to Lebesgue integration.)

Problem 3: Collaboration Allowed Let $X$ be an inner-product space and $T \in \mathcal{B}(X)$. Suppose $\langle Tx, x \rangle = 0$ for all $x \in X$.

(a) If $X$ is a complex inner product space, show that $T = 0$.

(b) If $X$ is a real inner product space, show that it is not necessarily true that $T = 0$ (Hint: consider rotations of the Euclidean plane).

Problem 4: Collaboration Allowed Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Prove that there exists a family of closed linear subspaces $\{\Omega_t \mid t \in [0,1]\}$ such that $\Omega_s$ is a strict subset of $\Omega_t$ whenever $s < t$. Note: this was the “hard” problem on the final exam in the fall 2005 class.