Due date: Friday, Nov. 14 2014, before 10 AM

Theme: Orthogonality in Hilbert Spaces, §6.2

Instructions: Problems marked with “Collaboration Allowed” mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to post requests for help on forums such as http://math.stackexchange.com/. On these problems, please write down the names of the students that you worked with.

On problems marked “No Collaboration,” collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Problem 1: No Collaboration. Problem 6.1 in the book: prove that a closed convex subset $C$ of a Hilbert space has a unique point of minimum norm.

Problem 2: Collaboration Allowed. Problem 6.2 in the book: Consider $C([0,1])$ with the sup-norm, and let $N$ be the closed linear space of all functions with zero mean, and $X$ the subspace of functions that have value 0 at 0), and define $M = N \cap X$.

(a) If $u \in C([0,1])$, prove $d(u, N) \overset{\text{def}}{=} \inf_{n \in N} \|u - n\| = |\bar{u}|$ where $\bar{u} = \int_0^1 u(x) \, dx$ is the mean of $u$.

(b) Show there is no unique projection onto $M$ by considering $u(x) \overset{\text{def}}{=} x$ and showing that $d(x, M) = \frac{1}{2}$ but that the infimum is not attained.

Problem 3: Collaboration Allowed. Problem 6.3 in the book: if $A$ is a subset of Hilbert space, prove that

(a) $A^\perp = \overline{A}^\perp$

(b) If $M$ is a linear subspace of the Hilbert space, then $M^{\perp \perp} = \overline{M}$.

Problem 4: Collaboration Allowed. You may use the internet for this problem. It is a “bonus problem” and will not be graded, so complete it if you like.

(a) Calculate $(a + ib)(c + id) = (ac - bd) + i(bc + ad)$ using only 3, not 4, multiplications, where $a, b, c, d \in \mathbb{R}$ and you may only multiply real numbers (for example, you are writing a computer program to multiply complex numbers, and you use the fact that your computer already knows how to multiply real numbers).

(b) Multiply the following two $2 \times 2$ matrices (entries are real-valued) using only 7 multiplication operations but unlimited addition operations:

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\cdot
\begin{pmatrix}
  e & f \\
  g & h
\end{pmatrix}
\]
(c) Conclude that, by generalizing the previous result to sub-blocks of matrices, and applying it recursively to matrices of size $N = 2^d$, that we can perform matrix multiplication in $O((7 + o(1))^d) = O(N^{\log_2 7 + o(1)}) \simeq O(N^{2.8})$ operations. This is the famous Strassen algorithm.