Problem #1 (9 points): Write the following complex numbers in polar exponential form:

(a) \( -4 \)
(b) \( -2 + 2i \)
(c) \( \sqrt{3} - i \)

Solution: Let \( n \) be an integer.

(a) \( r = 4 \) and \( \arg z = \pi \), so \( -4 = 4e^{i(\pi + 2n\pi)} \)
(b) \( r = 2\sqrt{2} \) and \( \arg z = 3\pi/4 \), so \(-2 + 2i = 2\sqrt{2}e^{i(3\pi/4 + 2n\pi)} \)
(c) \( r = 2 \) and \( \arg z = -\pi/6 \), so \( \sqrt{3} - i = 2e^{i(-\pi/6 + 2n\pi)} \)

Problem #2 (9 points): Write the following complex numbers in Cartesian, \( a + bi \), form:

(a) \( e^{5 - i\pi/2} \)
(b) \( \frac{1}{1 + i} \)
(c) \( \cos(c + i\pi/4) \), where \( c \) is real,
\[ \cos(z) = (e^{iz} + e^{-iz})/2, \text{ and } e^z = e^{x}e^{iy} \]

Solution:

(a) \( e^{5 - i\pi/2} = e^{5}e^{-i\pi/2} = 0 - ie^{5} \)
(b) \( \frac{1}{1 + i} = \frac{1 - i}{(1 + i)(1 - i)} = \frac{1 - i}{2} \)
(c) \[ \cos(c + i\pi/4) = \frac{1}{2} \left( e^{-\pi/4}e^{ic} + e^{\pi/4}e^{-ic} \right) \]
\[ = \frac{e^{-\pi/4} + e^{\pi/4}}{2} \cos c + \frac{i(e^{-\pi/4} - e^{\pi/4})}{2} \sin c \]
\[ = \cos c \cosh \frac{\pi}{4} - i \sin c \sinh \frac{\pi}{4} \]

Problem #3 (6 points): Show that
\[ z(z_{1} - z_{2}) + z_{1}(z_{2} - z) + z_{2}(z - z_{1}) = 0 \]
is an equation for a straight line through \( z_{1} \) and \( z_{2} \).

Solution: Let \( z = x + iy \), \( z_{1} = x_{1} + iy_{1} \), and \( z_{2} = x_{2} + iy_{2} \). Then
\[ (x + iy)[x_{1} - x_{2} + i(y_{2} - y_{1})] + (x_{1} + iy_{1})[x_{2} - x + i(y - y_{2})] \]
\[ + (x_{2} + iy_{2})[x - x_{1} + i(y_{1} - y)] \]
\[ = 2i(x_{1} - x_{2})y - 2i(y_{1} - y_{2})x + 2i(x_{2}y_{1} - x_{1}y_{2}) = 0 \]
and solving for \( y \) gives
\[ y = \frac{y_{2} - y_{1}}{x_{2} - x_{1}} \cdot \frac{x}{x} + \frac{x_{2}y_{1} - x_{1}y_{2}}{x_{2} - x_{1}} \]
which is the formula for a line going through \((x_{1}, y_{1})\) and \((x_{2}, y_{2})\).

Problem #4 (9 points): Establish the following results:

(a) \( \text{Re}(z) \leq |z| \)
(b) \( |z_{1}z_{2}| = |z_{1}| |z_{2}| \)
(c) \( |wz + \overline{w}z| \leq 2|wz| \)

Solution:

(a) For \( z = x + iy \), \( \text{Re}(z) = x \) and \( |z| = \sqrt{x^{2} + y^{2}} \).
Since \( y \) is real, \( \text{Re}(z) \leq |z| \).
(b) Let \( z_{1} = r_{1}e^{i\theta_{1}} \) and \( z_{2} = r_{2}e^{i\theta_{2}} \), so
\[ |z_{1}z_{2}| = |r_{1}r_{2}e^{i(\theta_{1} + \theta_{2})}| = r_{1}r_{2} \text{ and } |z_{1}| |z_{2}| = |r_{1}e^{i\theta_{1}}||r_{2}e^{i\theta_{2}}| = r_{1}r_{2} \]
(c) From the triangle inequality,
\[ |wz + \overline{w}z| \leq |w\overline{z}| + |\overline{w}z| \]. Since \( |z| = |\overline{z}| \) and
\[ |uv| = |u||v| \] (from [b]), \( |w\overline{z}| + |\overline{w}z| = 2|wz| \).
Thus, \( |wz + \overline{w}z| \leq 2|wz| \).

Problem #5 (18 points): Sketch the following regions and state if they are open, closed, bounded, compact, or connected.

(a) \( |2z + 1 + i| < 4 \)
(b) \( \text{Re}(z) \geq 4 \)
(c) \( |z| \leq |z + 1| \)
(d) \( 1 < |z - 1| \leq 2 \)
(e) \( 0 < \arg z \leq \pi/2 \)
(f) \( \text{Re}(z - z_{0}) > 0 \) and \( \text{Re}(z - z_{1}) < 0 \) for two complex numbers \( z_{0}, z_{1} \).

Solution:

(a) \( |2z + 1 + i| < 4 \) is a circle of radius 2 centered at \((-1 + i)/2\). This region is open, bounded, and connected.
Rez ≥ 4 ⇐⇒ x ≥ 4. This region is connected and unbounded.

|z| ≤ |z + 1| ⇐⇒ x^2 + y^2 ≤ (x + 1)^2 + y^2 ⇐⇒ 0 ≤ 2x + 1 ⇐⇒ x ≥ -1/2. This region is connected and unbounded.

1 < |z - 1| ≤ 2 ⇐⇒ |z - 1| ≤ 1 is an annulus centered at 1/2 with inner radius 1/2 and outer radius 1. This region is connected and bounded.

Re(1 - zn) = 1 + r cos θ + r^2 cos 2θ + \ldots + r^n cos nθ,

where z = r(cos θ + i sin θ).

Solution:

\[
\frac{1 - zn^{n+1}}{1 - z} = 1 + z + z^2 + z^3 + \ldots + z^n
\]

= 1 + re^{iθ} + r^2 e^{i2θ} + \ldots + r^n e^{inθ}

= \left(1 + r \cos θ + r^2 \cos 2θ + \ldots + r^n \cos nθ\right) + i\left(r \sin θ + r^2 \sin 2θ + \ldots + r^n \sin nθ\right).
So
\[ \Re \left( \frac{1-z^{n+1}}{1-z} \right) = 1 + r \cos \theta + r^2 \cos 2\theta + \cdots + r^n \cos n\theta. \]

**Problem #7 (8 points):** Use the series representation
\[ e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \quad |z| < \infty \]
to find representations for the following functions:
(a) \( \sin z \)
(b) \( \sinh z \)

**Solution:**
(a) Since \( \sin i^j - (-i)^j = 0 \) when \( j \) is even, we get that
\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i} \left( \sum_{j=0}^{\infty} \frac{(iz)^j}{j!} - \sum_{j=0}^{\infty} \frac{(-iz)^j}{j!} \right) = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}.
\]
(b) Since \( 1 - (-1)^j = 0 \) when \( j \) is even, we get that
\[
\sinh z = \frac{e^z - e^{-z}}{2} = \frac{1}{2} \left( \sum_{j=0}^{\infty} \frac{z^j}{j!} - \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{j!} \right) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}.
\]

**Problem #8 (8 points):** Use \( e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} e^{i\theta_2} \) to deduce
\[
\sin (u \pm v) = \sin u \cos v \pm \cos u \sin v, \\
\cos (u \pm v) = \cos u \cos v \mp \sin u \sin v.
\]

**Solution:** First, we note that
\[
e^{i(u \pm v)} = \cos(u \pm v) + i \sin(u \pm v).
\]

Then we use \( e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} e^{i\theta_2} \) on \( e^{i(u \pm v)} \) to get
\[
e^{i(u \pm v)} = e^{iu} e^{\pm iv} = (\cos u + i \sin u)(\cos \pm v \mp i \sin \pm v) = \cos u \cos v \mp \sin u \sin v \mp i(\sin u \cos v \pm \cos u \sin v).
\]
Taking the real part of both sides gives
\[
\cos (u \pm v) = \cos u \cos v \mp \sin u \sin v.
\]
Likewise, taking the imaginary parts give
\[
\sin (u \pm v) = \sin u \cos v \pm \cos u \sin v.
\]

**Problem #9 (8 points):** Discuss the following transformations (mappings) from the \( z \)-plane to the \( w \)-plane. E.g. which regions map to which, which points map to the same points?
(a) \( w = z^3 \)
(b) \( w = 1/z \)

**Solution:**
(a) For \( w = z^3 \), the wedges \( 0 \leq \arg z < 2\pi/3 \), \( 2\pi/3 \leq \arg z < 4\pi/3 \), and \( 4\pi/3 \leq \arg z < 2\pi \) each map to the whole \( w \)-plane. Indeed, \( z = r^{1/3} e^{i(\theta + 2m\pi)/3} \), \( m = 0, 1, 2 \), all map to \( w = re^{i\theta} \).
(b) For \( w = 1/z \), the area inside the unit circle maps to outside the unit circle and vice versa. Also, \( \arg z = -\arg w(z) \) modulo \( 2\pi \).

**Problem #10 (6 points):** Consider the transformation \( w = z + \frac{1}{z} \), where \( z = x + iy \) and \( w = u + iv \). Show that \( \mathcal{R} = \{ z : y > 0 \} \) maps to \( v > 0 \).

**Solution:** We note that \( \mathcal{R} = \{ r e^{i\theta} : r > 1 \text{ and } 0 < \theta < \pi \} \) and \( w = re^{i\theta} + \frac{e^{-i\theta}}{r} \).

Our usual method of showing this is to see what the boundaries of \( \mathcal{R} \) map to; so
\[
r = 1, 0 \leq \theta \leq \pi \quad \Rightarrow \quad w \in [-2,2],
\]
\[
r \geq 1, \theta = 0 \quad \Rightarrow \quad w = \frac{r^2 + 1}{r} \in [2,\infty),
\]
and
\[
r \geq 1, \theta = \pi \quad \Rightarrow \quad w = -\frac{r^2 + 1}{r} \in (-\infty,-2],
\]
which all map to the real \( w \)-line (\( v = 0 \)). Now, because
the transform is continuous, we just need to check one
point in \( \mathcal{R} \) to see if it maps to \( v > 0 \) or \( v < 0 \). Checking,
say, \( w(2i) = 2i + i/2 = 5i/2 \in \{w: v > 0\} \) gives that \( \mathcal{R} \)
maps to \( v > 0 \).

[If you checked \( |z| \to \infty, 0 < \theta < \pi \) – which wasn’t
required for full points – you might have used \( z = Re^{i\theta} \)
and found that \( w \sim Re^{i\theta} \) as \( R \to \infty \). And so found that
\( |z| \to \infty, 0 < \theta < \pi \) maps to \( |w| \to \infty, 0 < \theta < \pi \).]

Problem #11 (6 points): To what curves on the sphere
(in Fig. 1.2.6) do the lines \( \text{Re} z = x = 0 \) and \( \text{Im} z = y = 0 \)
correspond?

Solution:

(a) \( \text{Re} z = x = 0 \iff Y^2 + (Z - 1)^2 = 1 \), which is the
circle on the sphere in the \( YZ \)-plane.

(b) \( \text{Im} z = y = 0 \iff X^2 + (Z - 1)^2 = 1 \), which is the
circle on the sphere in the \( XZ \)-plane.

Extra-Credit Problem #12 (6 points): If \( |c| < 1 \), show
that the series
\[
\sum_{n=1}^{\infty} c^n e^{nz}
\]
converges absolutely for all values of \( z \).

Solution: Denote the \( n \)th term of the series as \( u_n \).
Then, if \( |c| < 1 \) and \( z \) is finite,
\[
\frac{u_{j+1}}{u_j} = \frac{c^{(n+1)^2-n^2} e^{z}}{c^{2n+1} e^{z}} \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus, by the ratio test, the series converges absolutely
for all finite \( z \).