Problem #1 (8 points): Use the definition of a limit to show the following:

(a) \( \lim_{z \to 0} \frac{z^2}{z} = 0. \)
(b) \( \lim_{z \to 1} [x + i(2x + y)] = 1 + i, \) where \( z = x + iy. \)

Problem #2 (16 points): Evaluate the following limits:

(a) \( \lim_{z \to z_0} z^{-m}, \) where \( m \) is an integer.
(b) \( \lim_{z \to \infty} \frac{z^2}{(3z + 1)^2}. \)
(c) \( \lim_{z \to 0} \frac{\text{Re}(z) \text{Im}(z)}{|z|}. \)

Problem #3 (8 points): Show that \( f(z) = z^{-2} \) is uniformly continuous in \( \frac{1}{z} < \text{Re} z < 1 \) but not in \( 0 < \text{Re} z < \frac{1}{2}. \)

Problem #4 (8 points): Prove or disprove:

\( \lim_{z \to 0} z \sin \left( \frac{1}{z} \right) = 0 \)

Problem #5 (8 points): Let \( f(z) \) be continuous and let \( \lim_{z \to 0} f(z) = 0. \) Show that

\( \lim_{z \to 0} \left( e^{f(z)} - 1 \right) = 0. \)

What can be said about

\( \lim_{z \to 0} \frac{e^{f(z)} - 1}{z}? \)

Problem #6 (6 points): Suppose we are given the following differential equations:

(a) \( \frac{d^3 w}{dt^3} - k^3 w = 0 \)
(b) \( \frac{d^6 w}{dt^6} - k^6 w = 0 \)

where \( t \) is real and \( k \) is a real constant. Find the general real solution of the above equations. Write the solution in terms of real variables.

Problem #7 (20 points): Where are the following functions differentiable? Compute \( f'(z) \) where it's defined.

(a) \( f(z) = \sin z. \)
(b) \( f(z) = \tan z. \)
(c) \( f(z) = [(z - 1)z]^{1/3}. \)
(d) \( f(z) = z \text{Re}(z). \)
(e) \( f(z) = x^2 + iy^2. \)

Problem #8 (8 points): Let \( f(z) \) denote the function

\[ f(z) = \begin{cases} \frac{z^2}{i} & z \neq 0, \\ 0 & z = 0. \end{cases} \]

Show that the \( f'(0) \) does not exist but \( u_x = v_y \) and \( v_x = -u_y \) at \( z = 0. \)

Problem #9 (18 points): Consider the differential equation,

\[ x^2 \frac{d^2 w}{dx^2} + x \frac{dw}{dx} + w = 0, \]

where \( x \) is real.

(a) Show that the transformation \( x = e^t \) implies that

\[ x \frac{d}{dx} = \frac{d}{dt}, \quad x^2 \frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt}. \]

(b) Use these results to find that \( w \) also satisfies the differential equation

\[ \frac{d^2 w}{dt^2} + w = 0. \]

(c) Use these results to establish that \( w \) has the real solution

\[ w = Ce^{i \log x} + \bar{C}e^{-i \log x} \]

or

\[ w = A \cos(\log x) + B \sin(\log x). \]

Extra-Credit Problem #10 (6 points): Find the sum of the series

\[ \sum_{n=0}^{\infty} a^n \cos(n\theta) \quad \text{and} \quad \sum_{n=0}^{\infty} a^n \sin(n\theta) \]

for \( a \in (-1, 1). \)

Extra-Credit Problem #11 (4 points): If \( f(z) \) is holomorphic and \( |f(z)| = (x^2 + y^2)^{\frac{1}{4}}, z = x + iy, \) find \( f(z). \)