Problem #1 (20 points): Find the branch points of the following functions and give a branch cut that will make each function single-valued:

(a) \((z - 1)^{-1/2}\)
(b) \((z + 1 - 2i)^{1/4}\)
(c) \(2\log z^2\)
(d) \(z^{\sqrt{2}}\)
(e) \(z^{1/3}(1 - z)^{2/3}\)

Problem #2 (10 points): Show that \(e^{\log z} = z\), but sometimes \(e^z \neq z\).

Problem #3 (16 points):

(a) Deduce the identity
\[
\tanh^{-1} z = \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right).
\]
(b) Use this identity to compute \(\frac{d}{dz} \tanh^{-1} z\).

Problem #4 (24 points):

(a) Show that the solution to Laplace's equation
\[
\nabla^2 T = \frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0
\]
in the region \(-\infty < u < \infty\) and \(v > 0\), with the boundary conditions
\[
T(u, v = 0) = \begin{cases} T_0, & u > 0 \\ -T_0, & u < 0 
\end{cases}
\]
is
\[
T(u, v) = T_0 \left( 1 - \frac{2}{\pi} \tan^{-1} \frac{v}{u} \right).
\]
(b) Now we'll use this result to solve Laplace's equation in \(|z| < 1\) with the boundary conditions
\[
T(r = 1, \theta) = \begin{cases} T_0, & 0 < \theta < \pi \\ -T_0, & \pi < \theta < 2\pi 
\end{cases}
\]
Show that
\[
w = i \left( \frac{1 - z}{1 + z} \right) \quad z = \frac{i - w}{i + w}
\]
maps
- \(|z| \leq 1\) to the upper-half \(w\)-plane \((w = u + iv\) and \(v \geq 0\)),
- \(r = 1, 0 < \theta < \pi\) onto \(v = 0, u > 0\), and
- \(r = 1, \pi < \theta < 2\pi\) onto \(v = 0, u < 0\).

(c) Use this mapping function to show that the solution of the boundary value problem in the circle is given by
\[
T(x, y) = T_0 \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{1 - (x^2 + y^2)^{1/2}}{2y} \right) \right]
\]
or, in polar coordinates,
\[
T(r, \theta) = T_0 \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{1 - r^2}{2r \sin \theta} \right) \right].
\]

Problem #5 (20 points): Let \(w(z) = (z^2 + 1)^{1/2}\), \((z - i) = r_1 e^{i\theta_1}\), and \((z + i) = r_2 e^{i\theta_2}\). Find which ranges for \(\theta_1\) and \(\theta_2\) would make the following branch cuts:

(a) \(\{i \kappa: \kappa \in \mathbb{R} \text{ and } |\kappa| < 1\}\) and
(b) \(\{i \kappa: \kappa \in \mathbb{R} \text{ and } |\kappa| > 1\}\).

Problem #6 (10 points): (Constructing harmonic functions) Let \(f(z) = u(x, y) + iv(x, y)\) be analytic in the open region \(R\). Assume that \(u^2 + v^2 \neq 0\) in \(R\). Show that
\[
\frac{uv_y - vu_y}{u^2 + v^2}
\]
is harmonic in \(R\). [Hint: If \(w(z)\) is analytic, is \(w'(z)/w(z)\) also analytic?]

Extra-Credit Problem #7 (10 points): Given the function \(f(z) = z^a(1 - z)^b\), what values of \(a\) and \(b\) guarantee \(z = 0\) is a branch point? What about for \(z = 1\)? Is \(z = z_\infty\)?