Homework 5
APPM 5450 Spring 2015 Applied Analysis 2

Due date: Friday, Feb. 20 2015, before 10 AM
Instructor: Dr. Becker
Theme: Weak convergence.

Instructions: Problems marked with “Collaboration Allowed” mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

There are no “No Collaboration” problems since these are replaced by an in-class quiz on Wednesday Feb 18 2015.

An arbitrary subset of these questions will be graded.

Problem 1: Collaboration Allowed Let $T$ be the left-shift operator on $\ell^2(\mathbb{N})$, i.e., $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Define the sequence $(T_n)_{n \in \mathbb{N}}$ by $T_n = T^n$, so $T_n$ is the “shift-$n$-to-the-left” operator. Answer the following and provide justification for your answer:

(a) Does $T_n \to 0$ uniformly?
(b) Does $T_n \to 0$ in the strong operator norm sense?
(c) Does $T_n \to 0$ in the weak operator norm sense?
(d) Does $T_n^* \to 0$ uniformly?
(e) Does $T_n^* \to 0$ in the strong operator norm sense?
(f) Does $T_n^* \to 0$ in the weak operator norm sense?
(g) Which of your answers above would change if we considered $\ell^2(\mathbb{Z})$ instead of $\ell^2(\mathbb{N})$? (No justification needed)

Problem 2: Collaboration Allowed Let $H = \ell^2(\mathbb{N})$ and $e_n$ be the canonical basis vectors. Which of the following sequences converge weakly? Which have strongly convergent subsequences, and which have weakly convergent subsequences?

(a) $x_n = n e_n$.
(b) $y_n = n^{-1/2} \sum_{j=1}^{\infty} e_j$.
(c) $x_n = e_n + e_m$ where $m = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$.

Problem 3: Collaboration Allowed If $U : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary, prove $\|U\| = 1$.

Problem 4: Collaboration Allowed Let $A$ denote a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $u$ be any element in $\mathcal{H}$ and set $u_n = e^{inA}u$. Prove that $(u_n)$ has a weakly convergent subsequence.

Problem 5: Collaboration Allowed Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and $U : \mathcal{H}_1 \to \mathcal{H}_2$ a unitary operator, and $A_1 \in B(\mathcal{H}_1)$ a self-adjoint operator. Define $A_2 \in B(\mathcal{H}_2)$ by $A_2 = UA_1U^{-1}$. Prove that $A_2$ is self-adjoint.

Problem 6: Collaboration Allowed Consider the Hilbert space $\mathcal{H} = L^2([-\pi, \pi])$ and the sequence of functions $\varphi_n(x) = x^2 \sin(nx)$. Does $(\varphi_n)$ converge strongly in $\mathcal{H}$? Does it converge weakly? If you answer yes to either question, what is the limit?