1. Show (theoretically) that the following initial value problem has a unique solution, and find the solution.

\[ y' = y \cos t, \quad 0 \leq t \leq 1, \quad y(0) = 1 \]

**Solution:** To show that the IVP has a unique solution for \(0 \leq t \leq 1\) we need to show that the function on the right-hand side of the equation is Lipschitz in \(y\). We have

\[
|f(t, y_1) - f(t, y_2)| = |y_1 \cos t - y_2 \cos t| \\
= |\cos t (y_1 - y_2)| \\
\leq |y_1 - y_2|
\]

So \(f(t, y)\) is Lipschitz in \(y\) with Lipschitz constant \(L = 1\)

The solution of the IVP is given as follows

\[
y' = y \cos t \quad \Rightarrow \quad \frac{y'}{y} = \cos t \quad \Rightarrow \quad \ln y = \sin t + A \\
\Rightarrow \quad y(t) = B e^{\sin t} \quad \Rightarrow \quad y(t) = e^{\sin t}
\]
2. Use one step of the Taylor series method (by hand [with a calculator]) of order $n = 1$, $n = 2$, and $n = 3$ to approximate the value of $y(0.01)$ where $y(t)$ is the solution to

$$y' = y^2 + ye^t, \quad y(0) = 1$$

\[ n = 1: \text{ We have} \]

$$y_{k+1} = y_k + hf(t_k, y_k) \implies y_{k+1} = y_k + hy'_k$$

where $y'_k = y_k^2 + ye^{t_k}$. For $k = 0$ we have $y'_0 = 1^2 + 1e^0 = 2$.

Then

$$y_1 = y_0 + hy'_0 = 1 + (0.01)(2) = 1.02$$

\[ n = 2: \text{ We have} \]

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2} y''_k \implies y_{k+1} = y_k + hy'_k + \frac{h^2}{2} y''_k$$

where here

$$y'' = \frac{d}{dt} (y^2 + ye^t) = 2yy' + y'e^t + ye^t \implies y''_k = 2(1)(2) + 2e^0 + 1e^0 = 7$$

Then

$$y_1 = y_0 + h(y_0^2 + y_0e^0) + \frac{h^2}{2} y''_0 = 1 + (0.01)(2) + \frac{0.01^2}{2} 7 = 1.2035$$

\[ n = 3: \text{ We have} \]

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2} y''_k + \frac{h^3}{6} y'''_k \implies y_{k+1} = y_k + hy'_k + \frac{h^2}{2} y''_k + \frac{h^3}{6} y'''_k$$

where here

$$y''' = 2\left[(y')^2 + yy''\right] + e^t \left[y'' + 2y' + y\right] \implies y'''_0 = 2(4 + 7) + (7 + 4 + 1) = 34$$

Then

$$y_1 = y_0 + h(y_0^2 + y_0e^0) + \frac{h^2}{2} y''_0 + \frac{h^3}{6} y'''_0 = 1 + (0.01)(2) + \frac{0.01^2}{2} 7 + \frac{0.01^3}{6} 34 \approx 1.020356$$
3. Consider the general initial value problem
\[ y' = f(t, y), \quad y(t_0) = y_0 \]

In class we derived Euler’s Method by expanding the solution to the IVP in a Taylor series. Another way to derive difference schemes is to write the solution over a length-h subinterval in the following integral form
\[ y(t + h) = y(t) + \int_t^{t+h} f(s, y(s)) \, ds \]

and discretize the equation using numerical quadrature.

(a) Derive the difference scheme that results from approximating the integral using the midpoint rule. Use Taylor series to find the local truncation error and the order of the method.

**Solution:** The midpoint quadrature rule is given by
\[ \int_t^{t+h} f(s, y(s)) \, ds \approx hf\left(t + \frac{h}{2}, y\left(t + \frac{h}{2}\right)\right) \]

The second argument in \( f \) is of course not computable exactly because we only know approximate values of \( y \) up to time \( t \). We approximate this value by taking a step of Euler’s method with step size \( h/2 \), giving
\[ \int_t^{t+h} f(s, y(s)) \, ds \approx hf\left(t + \frac{h}{2}, y\left(t + \frac{h}{2}\right) + \frac{h}{2} f(t, y(t))\right) \]

which then gives the midpoint difference scheme
\[ y_{k+1} = y_k + hf\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f(t_k, y_k)\right) \]

Expanding in a Taylor series centered at \( (t_k, y_k) \) and (in a total abuse of notation) denoting \( y_k' = y'(t_k) \), \( y_k'' = y''(t_k) \), etc, we have
\[ y_k + hy_k' + \frac{h^2}{2} y_k'' + O(h^3) = y_k + h \left[ f(t_k, y_k) + \frac{h}{2} f(t_k, y_k) + \frac{h}{2} f(t_k, y_k) f_y(t_k, y_k) + O(h^2) \right] \]

Next we notice that
\[ y''(t) = \frac{d}{dt} (y'(t)) = \frac{d}{dt} f(t, y(t)) = f_x(t, y(t)) + f_y(t, y(t)) y'(t) = f_x(t, y(t)) + f_y(t, y(t)) f(t, y(t)) \]

so the third term on the left side of the equation can be cancelled with the third and fourth terms on the right side of the equation. Finally, cancelling the first two terms on each side of the equation leaves us with a local truncation error of \( O(h^3) \). Since the local truncation error is \( O(h^3) \) the global error is \( O(h^2) \) and the midpoint method is second-order.
(b) Derive the difference scheme that results from approximating the integral using the trapezoidal rule. Use Taylor series to find the local truncation error and the order of the method.

**Solution:** The trapezoid quadrature rule is given by

\[ \int_{t}^{t+h} f(s, y(s)) \, ds \approx \frac{h}{2} [f(t, y(t)) + f(t + h, y(t + h))] \]

which gives the trapezoid difference scheme

\[ y(t + h) = y(t) + \frac{h}{2} (f(t, y(t)) + f(t + h, y(t + h))) \quad \text{or} \quad y_{k+1} = y_k + \frac{h}{2} [f(t_k, y_k) + f(t_{k+1}, y_{k+1})] \]

We find the local truncation error and order of the method by expanding the difference scheme with Taylor’s series. We can make our lives significantly easier here by noticing that

\[ f(t + h, y(t + h)) = y'(t + h) \]

which means we can expand this term using Taylor’s series for functions of a single variable. We then have

\[ y(t) + hy'(t) + \frac{h^2}{2} y''(t) + \frac{h^3}{6} y'''(t) + O(h^4) = y(t) + \frac{h}{2} \left[ y'(t) + y'(t) + hy''(t) + \frac{h^2}{2} y'''(t) + O(h^3) \right] \]

Cancelling terms on either side of the equation we’re left with

\[ \frac{h^3}{6} y'''(t) + O(h^4) = \frac{h^3}{4} y'''(t) + O(h^4) \]

Since the \( O(h^3) \) terms do not cancel we see that the local truncation error is \( O(h^3) \), the global error is \( O(h^2) \), and the trapezoid scheme is second-order.

(c) Describe the quadrature rule that leads to Euler’s Method.

**Solution:** The Forward Euler Method is \( y(t + h) = y(t) + hf(t, y(t)) \). The second term on the right-hand side comes from approximating the integral using the left-endpoint quadrature rule, i.e.

\[ \int_{t}^{t+h} f(s, y(s)) \, ds \approx hf(t, y(t)) \]
4. (a) Derive the modified Euler’s method

\[ y_{j+1} = y_j + hf\left(t_j + \frac{1}{2}h, y_j + \frac{1}{2}hf(t_j, y_j)\right) \]

by performing Richardson’s extrapolation on Euler’s method with step sizes \( h \) and \( h/2 \).

**Hint:** Assume the local truncation error in Euler’s method has the form \( Ch^2 \).

**Solution:** Richardson’s extrapolation is a general technique in numerical analysis that uses approximations on successive grid levels to obtain higher accuracy methods. One approximates the same value on grid levels \( h \) and \( h/2 \), and then combines the approximation in a clever way to cancel out the leading order error term.

Here we consider approximating \( y(t + h) \) using one step of Euler’s Method with step size \( h \), and two steps of Euler’s Method with step size \( h/2 \). In another abuse of notation we will denote \( t_{k+1/2} = t_k + \frac{h}{2} \) and \( y_{k+1/2} = y(t_k + \frac{h}{2}) \). We have

\[
\begin{align*}
y_{k+1} &= y_k + hf(t_k, y_k) + Ch^2 + O(h^3) \quad (1) \\
y_{k+1/2} &= y_k + \frac{h}{2}f(t_k, y_k) + C\frac{h^2}{4} + O(h^3) \\
y_{k+1} &= y_{k+1/2} + \frac{h}{2}f(t_{k+1/2}, y_{k+1/2}) + C\frac{h^2}{4} + O(h^3) \Rightarrow \\
y_{k+1} &= y_k + \frac{h}{2}f(t_k, y_k) + \frac{h}{2}f\left(t_{k+1/2}, y_k + \frac{h}{2}f(t_k, y_k)\right) + C\frac{h^2}{2} + O(h^3) \quad (2)
\end{align*}
\]

Looking at the form of the leading order error terms, we notice that if we multiply multiply (2) by 2 and subtract (1) the leading order terms will cancel. We have

\[
2y_{k+1} - y_{k+1} = 2[y_k + hf(t_k, y_k) + hf\left(t_{k+1/2}, y_k + \frac{h}{2}f(t_k, y_k)\right) + Ch^2] - [y_k + hf(t_k, y_k) + Ch^2] + O(h^3)
\]

Cancelling like terms we have

\[
y_{k+1} = y_k + hf\left(t_{k+1/2}, y_k + \frac{h}{2}f(t_k, y_k)\right) + O(h^3)
\]

which is Modified Euler.
(b) Solve the IVP in Problem 1 using modified Euler’s method and RK4 with time-step size $h = 0.1$. Plot the approximate resulting approximations along with the true solution on the same set of axes.

**Solution:** The following plot shows the approximations from the modified Euler method and RK4 vs the exact solution.

![Plot showing approximations from modified Euler and RK4 methods with the exact solution.]

(c) For each of the methods in (b), make a table of error ratios of the form $|e_h|/|e_{h/2}|$ where $e_h$ is the difference in the true solution and approximate solution at $t = 1$ using time steps $h = 2^{-\ell}$ for $\ell = 1, 2, \ldots$. Do the resulting ratios agree with the relevant global error analysis?

**Solution:** The following table shows the error ratios for RK4 and modified Euler with $h = 2^{-\ell}$ for $h = 1, 2, \ldots, 6$

<table>
<thead>
<tr>
<th>H</th>
<th>RK4</th>
<th>Modified Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>16.0072</td>
<td>2.79557</td>
</tr>
<tr>
<td>16.0891</td>
<td>3.37137</td>
<td></td>
</tr>
<tr>
<td>16.0693</td>
<td>3.67828</td>
<td></td>
</tr>
<tr>
<td>16.0414</td>
<td>3.83722</td>
<td></td>
</tr>
<tr>
<td>16.0233</td>
<td>3.91817</td>
<td></td>
</tr>
<tr>
<td>16.0242</td>
<td>3.95893</td>
<td></td>
</tr>
</tbody>
</table>

Note that the error ratios for RK4 are approaching $16 = 2^4$ and the error ratios for modified Euler are approaching $4 = 2^2$. This is consistent with the theory because RK4 is a fourth-order method and modified Euler is a second-order method.