This assignment is to help you prepare for the second midterm. It is not to be turned in.

1. Consider the following elliptic partial differential equation

\[-u_{xx} - u_{yy} = 2 \quad \text{in } \Omega = [0, 1]^2\]

\[u = 1 \quad \text{on } \partial \Omega\]

(a) Write down the difference scheme for the given problem using centered finite differences in space.

(b) Write down (explicitly) the matrix equation that results from discretizing the PDE on \([0, 1]^2\) where the mesh is broken into \(n = 4\) subintervals in each spatial direction.

(c) Discuss some numerical techniques you might use to solve the resulting linear system.

Solutions:

(a) Letting \(v_{ij} \approx u(x_i, x_j)\) we have

\[
\frac{1}{h^2} [-v_{i-1,j} + 2v_{ij} - v_{i+1,j}] + \frac{1}{h^2} [-v_{i,j-1} + 2v_{ij} - v_{i,j+1}] = 2 \quad \text{for } 1 \leq i, j \leq n - 1
\]

or

\[
\frac{1}{h^2} [-v_{i,j-1} - v_{i-1,j} + 4v_{ij} - v_{i+1,j} - v_{i,j+1}] = 2 \quad \text{for } 1 \leq i, j \leq n - 1
\]

(b) Since there are \(n = 4\) subintervals in each direction there are \(n+1 = 5\) grid points in each direction. But since the given problem has Dirichlet boundary conditions there are only \(n - 1 = 3\) interior points in each direction. Ordering the unknowns lexicographically in \(x\) and remembering to add the boundary data to the right-hand side vector, we have

\[
\begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
v_{11} \\
v_{21} \\
v_{31} \\
v_{12} \\
v_{22} \\
v_{32} \\
v_{13} \\
v_{23} \\
v_{33}
\end{bmatrix}
= \begin{bmatrix}
2 + 2/h^2 \\
2 + 1/h^2 \\
2 + h^2 \\
2 + 1/h^2 \\
2 \\
2 + 1/h^2 \\
2 + 2/h^2 \\
2 + 1/h^2 \\
2 + 2/h^2
\end{bmatrix}
\]

(c) There are many ways to solve the resulting linear system. The following table summarizes some of the possibilities

<table>
<thead>
<tr>
<th>Method</th>
<th>Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dense LU/Cholesky</td>
<td>Horribly Painfully Slow</td>
</tr>
<tr>
<td>Banded LU/Cholesky</td>
<td>Pretty Slow</td>
</tr>
<tr>
<td>Sparse LU/Cholesky</td>
<td>Pretty Slow</td>
</tr>
<tr>
<td>Jacobi, Gauss-Seidel, etc</td>
<td>Pretty Slow</td>
</tr>
<tr>
<td>Conjugate Gradient</td>
<td>Not Bad</td>
</tr>
<tr>
<td>Fast Fourier Transform</td>
<td>Pretty Fast</td>
</tr>
<tr>
<td>Multigrid</td>
<td>Wicked Fast</td>
</tr>
</tbody>
</table>
2. Consider the following wave equation

\[ u_{tt} - 4u_{xx} = 0 \quad \text{for} \quad 0 < x < 1 \]
\[ u(0, t) = u(1, t) = 0 \]
\[ u(x, 0) = \sin(\pi x) \]
\[ u_t(x, 0) = \cos(\pi x/2) \]

(a) Write down the explicit difference scheme that uses centered differences in both time and space.

(b) Show that the given difference scheme is second-order accurate in both time and space.

(c) The explicit difference scheme requires using an alternative method for computing the approximation at the first time-step. Explain how you could do this for the given problem without harming the second-order accuracy of the method.

(d) Use Von Neumann analysis to derive a stability condition on \( h \), and \( k \) for the explicit method.

Solutions:

(a) Letting \( v^n_\ell \approx u(x_\ell, t_n) \) we have

\[
\frac{1}{k^2} [v^{n-1}_\ell - 2v^n_\ell + v^{n+1}_\ell] - \frac{4}{h^2} [v^{n-1}_{\ell-1} - 2v^n_{\ell-1} + v^{n+1}_{\ell-1}] = 0
\]

Multiplying through by \( k^2 \) and defining \( \lambda = \frac{2k}{h} \) we have

\[
v^{n+1}_\ell = \lambda^2 v^{n}_{\ell-1} + 2 \left(1 - \lambda^2\right) v^n_\ell + \lambda^2 v^{n+1}_{\ell+1} - v^{n-1}_\ell
\]

(b) Expanding each term individually in Taylor’s series we have

\[
\frac{1}{k^2} [v^{n-1}_\ell - 2v^n_\ell + v^{n+1}_\ell] = \frac{1}{k^2} [u(x, t - k) - 2u(x, t) + u(x, t + k)]
\]

\[
= \frac{1}{k^2} \left\{ u(x, t) - ku_t(x, t) + \frac{k^2}{2} u_{tt}(x, t) - \frac{k^3}{6} u_{ttt}(x, t) + \frac{k^4}{24} u_{tttt}(x, t) + \mathcal{O}(k^5) \right\} - 2u(x, t) +
\]

\[
+ \left\{ u(x, t) + ku_t(x, t) + \frac{k^2}{2} u_{tt}(x, t) + \frac{k^3}{6} u_{ttt}(x, t) + \frac{k^4}{24} u_{tttt}(x, t) + \mathcal{O}(k^5) \right\}
\]

\[
= u_{tt} + \frac{k^2}{12} u_{tttt}(x, t) + \mathcal{O}(k^3)
\]

Similarly for the spatial derivatives we have

\[
\frac{1}{h^2} [v^{n-1}_{\ell-1} - 2v^n_{\ell-1} + v^{n+1}_{\ell-1}] = \frac{1}{h^2} [u(x - h, t) - 2u(x, t) + u(x + h, t)]
\]

\[
= \frac{1}{h^2} \left\{ u(x, t) - hu_x(x, t) + \frac{h^2}{2} u_{xx}(x, t) - \frac{h^3}{6} u_{xxx}(x, t) + \frac{h^4}{24} u_{xxxx}(x, t) + \mathcal{O}(h^5) \right\} - 2u(x, t) +
\]

\[
+ \left\{ u(x, t) + hu_x(x, t) + \frac{h^2}{2} u_{xx}(x, t) + \frac{h^3}{6} u_{xxx}(x, t) + \frac{h^4}{24} u_{xxxx}(x, t) + \mathcal{O}(h^5) \right\}
\]

\[
= u_{xx} + \frac{h^2}{12} u_{xxxx}(x, t) + \mathcal{O}(h^3)
\]
Combining these we have
\[
\frac{1}{k^2} \left[ v_{t-1}^{n-1} - 2v_{t}^{n} + v_{t+1}^{n+1} \right] - \frac{4}{h^2} \left[ v_{t-1}^{n} - 2v_{t}^{n} + v_{t+1}^{n+1} \right] = u_{tt} - 4u_{xx} + \frac{k^2}{12} u_{tttt}(x,t) - \frac{h^2}{3} u_{xxxx}(x,t) + O\left(k^3 + h^3\right)
\]

Since the leading terms in the remainder are \(O\left(k^2 + h^2\right)\) the given discretization is second-order accurate in both space and time.

(c) Expanding the solution at time \(t = k\) and at location \(x = x_\ell\) we have
\[
u(x_\ell, k) = u(x_\ell, 0) + ku_t(x_\ell, 0) + \frac{k^2}{2} u_{ttt}(x_\ell, 0) + O\left(k^3\right)
\]

Noting that \(u\) is the solution to the original PDE we have \(u_{ttt}(x_\ell, 0) = 4u_{xx}(x_\ell, 0)\) we have
\[
u(x_\ell, k) = u(x_\ell, 0) + ku_t(x_\ell, 0) + 2k^2 u_{xx}(x_\ell, 0) + O\left(k^3\right)
\]

Then using the initial data we have
\[
u(x_\ell, k) = \sin \left(\pi x_\ell\right) + k \cos \left(\pi x_\ell / 2\right) - 2k^2 \pi^2 \sin \left(\pi x_\ell\right) + O\left(k^3\right)
\]

Since this approximation has LTE \(O\left(k^3\right)\) we can use it for the first time step at each spatial location without worry about polluting the LTE of the second-order centered differences used for the remaining times in the simulation.

(d) Substituting an arbitrary Fourier mode of the form \(e^{\mu nk}e^{i\beta \ell h}\) into the difference scheme
\[
v_{\ell_{t+1}}^{n+1} = \lambda^2 v_{\ell_{t}}^{n} + 2 \left(1 - \lambda^2\right) v_{\ell_{t}}^{n} + \lambda^2 v_{\ell_{t+1}}^{n} - v_{\ell_{t-1}}^{n-1}
\]

we have
\[
e^{\mu(n+1)k}e^{i\beta \ell h} = \lambda^2 e^{\mu nk}e^{i\beta \ell h} + 2 \left(1 - \lambda^2\right) e^{\mu nk}e^{i\beta \ell h} + \lambda^2 e^{\mu nk}e^{i\beta \ell h} - e^{\mu(n-1)k}e^{i\beta \ell h}
\]

Factoring out and cancelling common terms we have
\[
e^{\mu nk} = \lambda^2 e^{-i\beta h} + 2 \left(1 - \lambda^2\right) e^{i\beta h} - e^{-\mu nk}
\]
\[
e^{\mu nk} + e^{-\mu nk} = 2 \left(1 - \lambda^2\right) + 2\lambda^2 \cos(\beta h) = 2 + 2\lambda^2 \left[\cos(\beta h) - 1\right] = 2 - 4\lambda^2 \sin^2(\beta h/2)
\]

Letting, for the moment, \(p = 2 - 4\lambda^2 \sin^2(\beta h/2)\) we have
\[
e^{\mu nk} + e^{-\mu nk} = p \Rightarrow \left(e^{\mu nk}\right)^2 - pe^{\mu nk} + 1 = 0
\]

This is a quadratic equation in \(e^{\mu nk}\), so using the quadratic formula we have
\[
e^{\mu nk} = \frac{p \pm \sqrt{p^2 - 4}}{2}
\]
We now consider the two cases when the discriminant is positive or negative.

If the discriminant is positive then we must have $p^2 > 4$. But we have $p = 2 - 4\lambda^2 \sin^2 (\beta h/2)$ so it can only be the case that $p < -2$. We then have for the smaller root

$$e^{\mu k} = \frac{p - \sqrt{p^2 - 4}}{2} < \frac{p}{2} < \frac{-2}{2} = -1$$

which leads to instability.

If, on the other hand, the discriminant is nonpositive (i.e. $p^2 \leq 4$) then we have

$$e^{\mu k} = \frac{p \pm i\sqrt{4 - p^2}}{2} \Rightarrow |e^{\mu k}|^2 = \frac{p^2 + (4 - p^2)}{4} = 1$$

which gives stability. In order for this to happen then we must have $-2 \leq p \leq 2$ which gives

$$-2 \leq 2 - 4\lambda^2 \sin^2 (\beta h/2) \leq 2 \Rightarrow 0 \leq \lambda^2 \sin^2 (\beta h/2) \leq 1 \Rightarrow \lambda \leq 1$$

which gives the explicit condition on $h$ and $k$ in this case as

$$\frac{2k}{h} \leq 1 \Rightarrow h \geq 2k$$
3. Consider the following advection equation

\[ u_t - 2u_x = 0 \quad \text{for} \quad 0 < x < 10 \]
\[ u(0, t) = u(10, t) \]
\[ u(x, 0) = e^{-(x-5)^2} \]

(a) Use the method of characteristics to find an exact solution to the given advection equation.

(b) Write down a difference scheme for the advection equation using Explicit Euler in time and up-winded differences in space.

(c) Use Von Nuemmann analysis to derive a stability condition on \( h \) and \( k \) for the given explicit up-winded scheme.

Solutions:

(a) We start by determining the characteristics. We have

\[ \frac{dx}{dt} = -2 \quad \Rightarrow \quad x(t) = -2t + x_0 \quad t = \frac{x_0 - x}{2} \]

Thus the characteristics are parallel straight lines in the \((x, t)\) plane with slope \(-1/2\). Along each characteristic the solution is constant, so we trace along a characteristic back to it’s starting value \(x_0\) at time \(t = 0\) and evaluate the initial condition there. Thus we have

\[ u(x, t) = g(x + 2t) = e^{-(x+2t-5)^2} \]

(b) Since the wave is traveling from right to left we need to use an upwinded difference scheme for the spatial derivative that get’s it’s information from the right. For this we use forward differences. We have

\[ \frac{v_{n+1}^\ell - v_n^\ell}{k} - 2 \left( \frac{v_{n+1}^\ell - v_{n}^\ell}{h} \right) = 0 \]

Letting \( \alpha = \frac{2k}{h} \) we have

\[ v_{n+1}^\ell = (1 - \alpha) v_n^\ell + \alpha v_{n}^\ell_{n+1} \]

(c) Substituting an arbitrary Fourier mode of the form \( e^{\mu nk} e^{i\beta \ell h} \) into the difference scheme we have

\[ e^{\mu(n+1)k} e^{i\beta \ell h} = (1 - \alpha) e^{\mu nk} e^{i\beta \ell h} + \alpha e^{\mu nk} e^{i\beta (\ell+1)h} \]

Factoring out and cancelling similar terms we have

\[ e^{\mu k} = (1 - \alpha) + \alpha e^{i\beta h} = (1 - \alpha) + \alpha [\cos (\beta h) + i \sin (\beta h)] \]

Then

\[ |e^{\mu k}|^2 = [(1 - \alpha) + \alpha \cos (\beta h)]^2 + \alpha^2 \sin^2 (\beta h) \]
\[ = (1 - \alpha)^2 + \alpha^2 + 2 \alpha (1 - \alpha) \cos (\beta h) \]
\[ = 1 - 2 \alpha (1 - \alpha) (1 - \cos (\beta h)) \]
\[ = 1 - 4 \alpha (1 - \alpha) (\sin^2 (\beta h/2)) \]

Then if \( 0 \leq \alpha \leq 1 \) we have

\[ |e^{\mu k}|^2 \leq 1 - \sin^2 (\beta h/2) \leq 1 \quad \text{then} \quad 0 \leq \alpha \leq 1 \quad \Rightarrow \quad \alpha \leq 1 \quad \Rightarrow \quad h \geq 2k \]