Starting at the origin, we move along the x-axis for 4 positive units, then we move downward 3 units. What are our coordinates now?

Solution:

(4, 0, -3). [REMEMBER – in Calc 3, we’re frequently dealing with 3 spatial dimensions, and conventionally, “down” and “up” refer to movements on the z-axis.]

10.1 #4 Sketch planar projections of (2, 3, 5).

Solution:

(0, 0, 5) – yz projection.

(0, 3, 5) – xz projection.

(2, 0, 5) – xy projection.

Length of box diagonal

Length of box diagonal

\[= \sqrt{(2-0)^2 + (2-0)^2 + (5-0)^2} \]

\[= \sqrt{38}.\]
Find the side lengths of these triangles, and characterize their shape.

a) P(3, -2, -3), Q(7, 0, 1), R(1, 2, 1).

Solution: We analyze each edge in turn, then compare them to one another.

Length $\overline{PQ} = \sqrt{(7-3)^2 + (0-(-2))^2 + (1-(-3))^2} = \sqrt{36} = 6$

Length $\overline{QR} = \sqrt{(1-7)^2 + (2-0)^2 + (1-1)^2} = \sqrt{40}$

Length $\overline{RP} = \sqrt{(3-1)^2 + (-2-2)^2 + (-3-1)^2} = \sqrt{36} = 6$.

Two of the side lengths are equal, so the triangle is isosceles. The squares of no two side lengths equal the square of the remaining side length, so the triangle is not a right triangle.
b) Characterize the following triangle:
P(2,1,0), Q(4,1,1), R(4,-5,4).

Solution: As in part a),

Length \( \overrightarrow{PQ} \) = \( \sqrt{(4-2)^2 + (1-1)^2 + (1-0)^2} = \sqrt{9} = 3 \)

Length \( \overrightarrow{QR} \) = \( \sqrt{(4-4)^2 + (-5-1)^2 + (4-1)^2} = \sqrt{45} = 3\sqrt{5} \)

Length \( \overrightarrow{RP} \) = \( \sqrt{(2-4)^2 + (-1+5)^2 + (0-4)^2} = \sqrt{36} = 6 \)

No two side lengths of \( \triangle PQR \) are equal, so the triangle is neither isosceles nor equilateral.

However, the square of length \( \overrightarrow{PQ} \) plus the square of length \( \overrightarrow{RP} \) equals the square of length \( \overrightarrow{QR} \), so \( \triangle PQR \) is a right triangle with hypotenuse \( \overrightarrow{QR} \).
Find the distances of the following structures to \((4, -2, 6)\).

a) The \(xy\)-plane. Solution: The \(z\)-coordinate is \(6\).

b) The \(yz\)-plane. Solution: The \(x\)-coordinate is \(4\).

c) The \(xz\)-plane. Solution: Magnitude of \(y\)-coord, \(\sqrt{2}\).

d) The \(x\)-axis.

Solution: The projection of \((4, -2, 6)\) onto the \(x\)-axis is \((4, 0, 0)\). The distance between \((4, -2, 6)\) and \((4, 0, 0)\) is \(\sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}\).

e) The \(y\)-axis.

Solution: As in part d), the projection of \((4, -2, 6)\) onto the \(y\)-axis is \((0, -2, 0)\). The distance in question is \(\sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}\).

f) The \(z\)-axis. Solution: As above, the projection is \((0, 0, 6)\) and distance is \(\sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}\).
Determine whether the following points lie on a straight line.

(a) A(2, 4, 2), B(3, 7, -2), C(1, 3, 3).

Solution: The points lie on a straight line if and only if the sum of two side lengths of the triangle connecting the points equals the remaining side length (not their squares).

Length $\overline{AB} = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}$

Length $\overline{BC} = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{45} = 3\sqrt{5}$

Length $\overline{CA} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

It's apparent that no two side lengths equal the sum of differences. Therefore, the points are not on a line.
b) As with a), using B(0, -5, 5), E(1, 2, 4), F(3, 4, 2).

Solution: \[ \text{Length } \overrightarrow{DE} = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11} \]

\[ \text{Length } \overrightarrow{EF} = \sqrt{2^2 + 6^2 + 2^2} = \sqrt{44} = 2\sqrt{11} \]

\[ \text{Length } \overrightarrow{FD} = \sqrt{3^2 + 9^2 + 3^2} = \sqrt{99} = 3\sqrt{11} \]

Since \((\text{Length } \overrightarrow{DE})^2 + (\text{Length } \overrightarrow{EF})^2 = (\text{Length } \overrightarrow{FD})^2\) (20)

The points lie on a + line (in 3 dimensions).

10.1 #20 Find an equation of the largest sphere with center \((5, 4, 9)\) contained in the first octant (positive \(x\), \(y\), and \(z\) values)

Solution: The radius of said sphere must be no larger than the distance to the nearest coordinate plane. In this case, that distance is \(4\) (to \(x\)-plane). The equation for such a sphere is:

\[ (x-5)^2 + (y-4)^2 + (z-9)^2 \leq 16. \]
Find an equation of the set of all points equidistant from $A(-1,5,3)$ and $B(6,2,-2)$.

**Solution:** Whenever we’re equating distances, it’s often simpler to equate squared distances algebraically. A point $(x,y,z)$ that satisfies our condition satisfies the following equation:

$$(x+1)^2 + (y-5)^2 + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z-(-2))^2$$

$$(x^2+2x+1) + (y^2-10y+25) + (z^2-6z+9) = (x^2-12x+36) + (y^2-4y+4) + (z^2+4z+4)$$

$$14x - 6y - 10z - 9 = 0$$

$$14x - 6(y+3z) - 10z = 0 \quad \boxed{\text{similar}}$$

With linear $x$, $y$, and $z$ terms, we know this equation is that of a plane.

(In fact, it can be interpreted as a plane centered at $(0,-3z,0)$ with normal vector $(14,-6,-10)$, as you’ll learn later in the course.)
10.2 # 13

Find \( \vec{a} + \vec{b} \), \( 2\vec{a} + 3\vec{b} \), \(|\vec{a}|\), and \(|\vec{a} - \vec{b}|\).

Solution:

\( \vec{a} + \vec{b} = \langle 5 + (-3), -12 + (-6) \rangle = \langle 2, -18 \rangle \)

\( 2\vec{a} + 3\vec{b} = \langle 2(5) + 3(-3), 2(-12) + 3(-6) \rangle = \langle 1, -42 \rangle \)

\(|\vec{a}| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13 \)

\(|\vec{a} - \vec{b}| = \sqrt{(5 - (-3))^2 + (-12 - (-6))^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10 \)

10.2 # 16

As in #13, with \( \vec{a} = \langle 2, -4, 4 \rangle \) and \( \vec{b} = \langle 0, 2, -1 \rangle \)

Solution:

\( \vec{a} + \vec{b} = \langle 2 + 0, -4 + 2, 4 + (-1) \rangle = \langle 2, -2, 3 \rangle \)

(or \( 2\hat{i} - 2\hat{j} + 3\hat{k} \))

\( 2\vec{a} + 3\vec{b} = \langle 2(2) + 3(0), 2(-4) + 3(2), 2(4) + 3(-1) \rangle = \langle 4, -2, 5 \rangle \)

\(|\vec{a}| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{36} = 6 \)

\(|\vec{a} - \vec{b}| = \sqrt{(2-0)^2 + (-4-2)^2 + (4-(-1))^2} = \sqrt{2^2 + 6^2 + 5^2} = \sqrt{65} \)
10.2 # 22

A child pulls a sled through the snow as shown below. Find horizontal and vertical components of the force \( \vec{F} \), \( (50 \text{ N}) \) \( \text{(child)} \)

Solution:

(choosen coordinate system)

We can find the magnitudes of \( F_x \) and \( F_y \) by using this triangle:

\[
|F_x| = 50 \cos(38^\circ), \quad |F_y| = 50 \sin(38^\circ)
\]
Ropes hold up a 5 kg mass as shown below.
Find the tension in each wire and the magnitude of the tensions.

\[ \text{Solution:} \]

We know two things:

1) The vertical components must sum to the force of gravity on the star \( (\vec{A}_z + \vec{B}_z = (9.8 \times 5) \text{ N}) \)

2) The horizontal components must sum to zero so that the star is stationary \( (\vec{A}_x + \vec{B}_x = 0) \).

Using trigonometric relationships, we have

\[ \sin(52^\circ) |\vec{A}| + \sin(40^\circ) |\vec{B}| = 49 \text{ N} \]

\[ -\cos(52^\circ) |\vec{A}| + \cos(40^\circ) |\vec{B}| = 0 \text{ N}, \]

\( \text{negative x-direction} \)
Use your favorite method for solving systems of equations (substitution, equation arithmetic, MATLAB - my personal poison of choice) to find that $|A| \approx 37.56$, $|B| \approx 30.19$. (N)

With this knowledge,

$$\hat{A} = \langle \cos(52^\circ) | A \rangle, \sin(52^\circ) | A \rangle \simeq \langle -23.1, 29.6 \rangle.$$  

$$\hat{B} = \langle \cos(40^\circ) | B \rangle, \sin(40^\circ) | B \rangle \simeq \langle 23.1, 19.4 \rangle.$$
10.3 # 33

Show that the vector \( \text{orth}_a \mathbf{b} = \mathbf{b} - \text{proj}_a \mathbf{b} \)

is orthogonal to \( \mathbf{a} \).

**Solution**

\[
\text{orth}_a \mathbf{b} = \mathbf{b} - \text{proj}_a \mathbf{b} = \mathbf{b} - \left( \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \right) \mathbf{a}
\]

\[
= \mathbf{b} - \left( \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \right) \mathbf{a}
\]

\[
\mathbf{a} \cdot \text{orth}_a \mathbf{b} = \mathbf{a} \cdot \left( \mathbf{b} - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right) \mathbf{a} \right)
\]

\[
= (\mathbf{a} \cdot \mathbf{b}) - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right) \|\mathbf{a}\|^2
\]

\[
= (\mathbf{a} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b}) = 0.
\]

10.3 # 44 Find the angle between a diagonal of a cube and a diagonal of a face.

**Solution:** We use the unit cube for simplicity. The angle in question \( \theta \) is \( \sin^{-1} \left( \frac{1}{\sqrt{3}} \right) \), or about \( 35.3^\circ \) (0.616 rad).
Suppose all sides of a quadrilateral are equal in length, with sides parallel. Show that the diagonals are perpendicular.

Solution: See the drawing at right.

We have constructed such a parallelogram, with $|\vec{a}| = |\vec{b}|$.

Now observe the dot product of the two diagonals:

\[
(\vec{b} - \vec{a}) \cdot (\vec{a} + \vec{b}) = (\vec{b} \cdot \vec{a}) - (\vec{a} \cdot \vec{a}) + (\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{a})
\]

\[
= (\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{a})
\]

\[
= |\vec{b}|^2 - |\vec{a}|^2 = 0
\]

(since $|\vec{b}| = |\vec{a}|$).
10.3 # 50

a) Give a geometric interpretation of the triangle inequality \( |\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}| \).

Solution:

What the triangle inequality says is that under no circumstances may the sum of two vectors yield a vector with a magnitude greater than the sum of the original two magnitudes.

One way to think about this is that for two magnitudes \( |\vec{a}| \) and \( |\vec{b}| \) of vectors \( \vec{a} \) and \( \vec{b} \), the magnitude of \( \vec{c} = \vec{a} + \vec{b} \) is maximized when \( \vec{a} \) and \( \vec{b} \) point in the same direction.

In this case, \( |\vec{c}| = |\vec{a}| + |\vec{b}| \).

(\( \vec{c} \), \( \vec{d} \), \( \vec{e} \))

(\( \vec{f} \), \( \vec{g} \), \( \vec{h} \)).
10.3 # 50 (cont'd)

b) Use Cauchy-Schwarz to prove the triangle inequality.

**Solution:**

\[
|a + b|^2 = (a + b) \cdot (a + b) = |a|^2 + |b|^2 + 2(a \cdot b)
\]

\[
\leq |a|^2 + |b|^2 + 2|a||b| \quad \text{(from C-S)}
\]

\[
= (|a|^2 + |b|^2)^{\frac{1}{2}}
\]

So,

\[
|a + b| \leq |a| + |b|.
\]