Exam #3
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INSTRUCTIONS: On the front of your bluebook please print your name, student ID, date and course code. Show all your work in your bluebook. Please start each new problem on a new page. Solve the problems in the same order as they are requested. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.

1. (10 points.) Consider a time-homogeneous Markov process over a finite state space $S$ and rate matrix $Q$ such that: $\sum_{j: j \neq i} Q(i, j) = \sum_{j: j \neq i} Q(j, i)$, for each state $i$ in $S$. Guess a stationary distribution of the process and show that your guess is correct.

2. (20 points.) Customers arrive at a barbershop at rate $\lambda > 0$ and are served by its only barber at rate $2\mu > 0$ (note the factor of 2). Unfortunately, customers waiting in line become impatient and leave at rate $\mu$, regardless of their position in the queue. Let $X = (X_t)_{t \geq 0}$ denote the number of customers in the barbershop at time $t$.

(a) Represent the rate matrix of $X$ as a directed graph with weighted edges.
(b) Does $X$ have a stationary distribution? If so, determine it explicitly.
(c) Under what conditions (if any) is $X$ time reversible? Explain briefly.
(c) In equilibrium, what's the output processes from the barbershop? Explain briefly.

3. (30 points.) Let $N \geq 0$ denote a fixed integer. Consider an M/M/1-queue in which customers enter the queue only when there are less than $N$ people waiting in line. Let $X = (X_t)_{t \geq 0}$ denote the number of people in the queue (i.e. waiting or undergoing service) at time $t$. Assuming that arrivals occur at rate $\lambda > 0$ and customers are served at rate $\mu > 0$, with $\lambda \neq \mu$, respond:

(a) Under what conditions does $X$ have a stationary distribution? When these conditions are satisfied, what's the stationary distribution of $X$?
(b) In the long run, what fraction of customers enter the queue? You do not need to justify your answer!
(c) Assuming that rates are measured in customers-per-hour units, on average, how many customers are attended by the teller in a 4 hour shift? You do not need to justify your answer!

(ONE MORE PROBLEM ON THE BACK!)
4. (40 points.) Consider constants $0 < p_1, p_2, p_3 < 1$ and the network of queues with migration as in the figure below. As usual, $\lambda_i > 0$ denotes the external arrival rate to queue-$i$, and $p(i, j)$ is the probability that a departure from queue-$i$ enters queue-$j$. Assume that $p(1, 2) = p_1$, $p(2, 3) = p_2$ and $p(3, 1) = p_3$, and that $p(i, j) = 0$ otherwise.

![Diagram of queue network]

Based on the above information, respond:

(a) Does the above network satisfy Hypothesis A? Explain briefly.

(b) Based on your answer to part (a), what can you say about the vector $\vec{r}$ of asymptotic output rates from each queue?

(c) Determine $\vec{r}$ explicitly.
   **Hint:** Determine $r_1$ first and then exploit the symmetry of the network to guess $r_2$ and $r_3$.

(d) Suppose that the network is composed only by M/M/1-queues and that each server takes care of customers at rate $\mu$. Under what condition on the parameters $(\lambda_1, \lambda_2, \lambda_3, \mu, p_1, p_2, p_3)$ does this network satisfy Hypothesis B?
   **Hint:** Focus on queue-1 and guess the additional conditions implied by the other queues.

(e) If the condition in part (d) is satisfied, what is the stationary distribution of the network?

**DURATION: 90 MINUTES**
The condition on $Q$ means that flow out of each state is the same as flow in, thus $\Pi(i) = \frac{1}{|S|}$ for every $i \in S$, is a natural guess. To double-check:

$$\Pi(i) = \sum_{j \neq i} \frac{1}{|S|} Q(j,i) = \frac{1}{|S|} \left( \sum_{j \neq i} Q(j,i) \right) = \frac{1}{|S|} \left( \sum_{j \neq i} Q(i,j) + \sum_{j \neq i} Q(j,i) \right) = 0 \quad \text{i.e. } \Pi \text{ is stationary.}$$

\[\text{Diagram:}\]

\[\text{Nodes:} 0, 1, 2, \ldots, 2, 2, \ldots\]

\[\text{Edges:} 0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 3, \ldots, 2 \rightarrow 2, 2 \rightarrow 3, \ldots\]

\[\text{Transition rates:} 2\mu, 3\mu, 4\mu, \ldots, (2n+1)\mu, (2n+2)\mu, \ldots\]

\[\text{Using DB-condition, consider:}\]

$$\nu(t) = e^t - 1 \quad \nu(t) = \begin{pmatrix} (2\mu)^t \\ \vdots \\ (2\mu)^t \end{pmatrix} \quad \nu(0) = \begin{pmatrix} e^{2\mu} - 1 \\ \vdots \\ e^{2\mu} - 1 \end{pmatrix}$$

To have a stationary distribution need $\frac{\nu(t)}{\nu(0)} = 1$ i.e.

$$\nu(t) \sum_{i=0}^{\infty} \frac{(2\mu)^i}{(2\mu)^i} = \nu(0) \sum_{i=1}^{\infty} \frac{(2\mu)^i}{(2\mu)^i} = \nu(0) \left( \frac{e^{2\mu}}{e^{2\mu} - 1} \right)$$

i.e. need $\nu(0) = \frac{(2\mu)^t}{e^{2\mu} - 1}$

So $\Pi(i) = \frac{(2\mu)^{i+1}}{(i+1)! \cdot (e^{2\mu} - 1)}$, for $i \geq 0$. 


(c) Since $\Pi$ satisfies the DB-condition, $X$ is time-reversible when it is stationary.

(d) When $X$ is in equilibrium the output process has the same distribution as the input $b/c$ $X$ is time-reversible. In particular, the output process is an HPP($\lambda$).

P31 (a) $X$ is a birth-death chain with rate matrix:

\[
\begin{array}{cccccccc}
& 0 & 1 & 2 & \cdots & N \\
\mu & 0 & \mu & \mu & \cdots & \mu \\
\end{array}
\]

so by detailed balance: $\Pi(i) = c \cdot \frac{\lambda^i}{\mu^i} = c \cdot (\frac{\lambda}{\mu})^i, \ i = 0: N$

where

\[
c = \frac{1}{\sum_{i=0}^{\infty} (\frac{\lambda}{\mu})^i} = \frac{(\frac{\lambda}{\mu})^{i+1} - 1}{(\frac{\lambda}{\mu})^{i+1} - 1}
\]

i.e.

\[
\Pi(i) = \frac{(\frac{\lambda}{\mu})^{i+1} - 1}{(\frac{\lambda}{\mu})^{i+1} - 4}, \ i = 0: N
\]

regardless of $\lambda$ and $\mu$ as long as $\lambda \neq \mu$.

(b) $= (1 - \Pi(N))$

(c) $= \frac{4 \cdot (1 - \Pi(0))}{1/\mu} = 4 \mu (1 - \Pi(0))$
(a) Yes \( b/c \) from each queue there is a positive prob. of exiting the system since \( P_0 < 1 \) i.e. \( q(x) > 0 \)

(b) That is well-defined and unique \( b/c \) the matrix \((I - P)\) is invertible.

(c) \( \pi \) solves the linear system:

1. \( \pi_1 = \lambda_1 + p_3 \pi_3 \)
2. \( \pi_2 = \lambda_2 + p_1 \pi_1 \)
3. \( \pi_3 = \lambda_3 + p_2 \pi_2 \)

Replace (3) in (1):
\[
\pi_1 = \lambda_1 + p_3 (\pi_3 + p_2 \pi_2)
\]

Replace (2) into this:
\[
\pi_1 = \lambda_1 + p_3 \lambda_3 + p_2 p_3 (\pi_2 + p_1 \pi_1)
\]

Solving for \( \pi_1 \) we obtain:
\[
\pi_1 = \frac{\lambda_1 + p_3 \lambda_3 + p_2 p_3 \lambda_2}{1 - p_1 p_2 p_3}
\]

By symmetry, we obtain:
\[
\pi_2 = \frac{\lambda_2 + p_1 \lambda_1 + p_1 p_3 \lambda_3}{1 - p_1 p_2 p_3}
\]
\[
\pi_3 = \frac{\lambda_3 + p_2 \lambda_2 + p_1 p_2 \lambda_1}{1 - p_1 p_2 p_3}
\]
(d) In queue-1 need \( \sum_{i=0}^{\infty} \frac{pi^i}{\mu_i} = \sum_{i=0}^{\infty} \left( \frac{\pi_i}{\mu} \right)^i < \infty \) i.e. need \( \frac{\pi_1}{\mu} < 1 \).

By symmetry, we also need \( \frac{\pi_2}{\mu} < 1 \) and \( \frac{\pi_3}{\mu} < 1 \).

e) Based on the above, the marginal stat. dist. of queue-1 is geometric \( (1 - \frac{\pi_1}{\mu}) \). Thus, by symmetry:

\[
\Pi(i, j, k) = (1 - \frac{\pi_1}{\mu}) \left( \frac{\pi_1}{\mu} \right)^i \cdot (1 - \frac{\pi_2}{\mu}) \left( \frac{\pi_2}{\mu} \right)^i \cdot (1 - \frac{\pi_3}{\mu}) \left( \frac{\pi_3}{\mu} \right)^k,
\]

for each \( i, j, k \geq 0 \).