**Theorem:** Let $f : \mathbb{X} \to \mathbb{Y}$ be a continuous function. Let $K \subseteq \mathbb{X}$ be compact. Then $f$ is uniformly continuous on $K$.

**Proof:**

- Let $\varepsilon > 0$. We need to find a $\delta(\varepsilon)$ so that
  $$d_X(x, y) < \delta(\varepsilon) \Rightarrow d_Y(f(x), f(y)) < \varepsilon \; \forall \; x, y \in K.$$
- $f$ continuous $\Rightarrow$ for every $y \in \mathbb{X}$ there exists a $\delta(y, \varepsilon)$ such that
  $$d_X(x, y) < \delta(y, \varepsilon) \Rightarrow d_Y(f(x), f(y)) < \varepsilon \; \forall \; x \in \mathbb{X}.$$
- Note that
  $$K \subseteq \bigcup_{y \in K} B_{\frac{1}{2}\delta(y, \varepsilon/2)}(y).$$
  (This is one of those places where I would start with $B_{\delta(y, \varepsilon)}$ but then realize I had to go back and put in some kind of fractions.)
- Since these balls from an open cover of $K$, $K$ compact implies there exists a finite subcover
  $$K \subseteq \bigcup_{i=1}^n B_{\frac{1}{2}\delta(y_i, \varepsilon/2)}(y_i).$$
- Take $\delta(\varepsilon) = \min\{\frac{1}{2}\delta(y_1, \varepsilon/2), \frac{1}{2}\delta(y_2, \varepsilon/2), \ldots, \frac{1}{2}\delta(y_n, \varepsilon/2)\}$.
- Now for any $x, z \in K$ with $d_X(x, z) < \delta(\varepsilon)$, we will show that $d_Y(f(x), f(z)) < \varepsilon$.
  - $x \in K \Rightarrow \exists \; j \in \{1, 2, \ldots, n\}$ such that $x \in B_{\frac{1}{2}\delta(y_j, \varepsilon/2)}$.
    So, $d_X(x, y_j) < \frac{1}{2}\delta(y_j, \varepsilon/2)$.
  - Also, $d_X(x, z) < \delta(\varepsilon) \leq \frac{1}{2}\delta(y_j, \varepsilon/2)$ since $\delta(\varepsilon)$ was chosen to be the minimum of the $\frac{1}{2}\delta(y_i, \varepsilon/2)$ and therefore is less than the specific $\frac{1}{2}\delta(y_j, \varepsilon/2)$.
  - So,
    $$d_X(z, y_j) \leq d_X(z, x) + d_X(x, y_j) < \frac{1}{2}\delta(y_j, \varepsilon/2) + \frac{1}{2}\delta(y_j, \varepsilon/2) = \delta(y_j, \varepsilon/2).$$
  - Since
    $$d_X(x, y_j) < \frac{1}{2}\delta(y_j, \varepsilon/2) < \delta(y_j, \varepsilon/2),$$
    both $x$ and $z$ are within $\delta(y_j, \varepsilon/2)$ of $y_j$.
  - By continuity of $f$ (see second bullet point), we then know that
    $$d_Y(f(x), f(y_j)) < \varepsilon/2 \; \text{ and } \; d_Y(f(z), f(y_j)) < \varepsilon/2.$$
  - So,
    $$d_Y(f(x), f(z)) \leq d_Y(f(x), f(y_j)) + d_Y(f(z), f(y_j)) < \varepsilon/2 + \varepsilon/2 < \varepsilon,$$
    as desired! \qed