Solution: APPM 3310: Matrix Methods — Exam #1 — Fall 2011

(1) (30 pts) **Always True** or **False** questions. Answer **Always True** or **False**, and in each case you must also give a short **explanation** of your answer.

(a) (7 pts) If $F$ and $M$ are matrices such that $FM = I$ then $MF = I$.
(b) (7 pts) If $K$ and $B$ are nonsingular $n \times n$ matrices then $(K + B)^{-1} = B^{-1} + K^{-1}$.
(c) (8 pts) If $A$ is a $3 \times 3$ matrix with $LU$-decomposition $A = LU$, then $\det(A) = (-1)^3 \det(U)$.
(d) (8 pts) Let $S \subset M_{m\times n}$ be the set of matrices of rank 2, then $S$ is a vector subspace of $M_{m\times n}$.

**Solution:**

(a) False, consider the counterexample $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, them $FM \neq I$.

(b) False, consider $K = I$ and $B = -I$ then $K$ and $B$ are invertible but $K + B = O$ is not.

(c) False, if $A$ is square then $\det(A) = \det(U)$.

(d) False, closure fails, for example the zero matrix $O$ has rank 0 and therefore is not in $S$.

(2) (35 pts) Consider the system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & 2 & 1 \\ -1 & -2 & 7 & 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is an arbitrary vector.

(a) (8 pts) Use Gaussian elimination to transform the augmented matrix $(A|\mathbf{b})$ into row echelon form.

(b) (8 pts) Find the $LU$-decomposition of $A$.

(c) (2 pts) What is $\text{rank}(A)$?

(d) (7 pts) Find a basis for the set $\{\mathbf{x} \in \mathbb{R}^4 | A\mathbf{x} = \mathbf{0}\}$.

(e) (10 pts) Let $\mathbf{b} = (0, 3, 2k)^T$. For what values of $k$ does the system $A\mathbf{x} = \mathbf{b}$ have no solution? infinitely many solutions? a unique solution?

**Solution:**

(a) Note,

$$(A|\mathbf{b}) = \begin{pmatrix} 1 & 2 & 3 & 4 & b_1 \\ -1 & -2 & 2 & 1 & b_2 \\ -1 & -2 & 7 & 6 & b_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 5 & 5 & b_1 + b_2 \\ 0 & 0 & 10 & 10 & b_1 + b_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 5 & 5 & b_1 + b_2 \\ 0 & 0 & 0 & 0 & -b_1 - 2b_2 + b_3 \end{pmatrix}$$

(b) $A = LU$ where $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(c) From part (b), $\text{rank}(A) = 2$.

(d) Solving $A\mathbf{x} = \mathbf{0}$ from part (a) yields $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and so

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} w$$

and so the basis is $\{(-2, 1, 0, 0)^T, (-1, 0, -1, 1)^T\}$.
(e) From part (a) \((A|b) = \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 5 & 5 & 3 \\ 0 & 0 & 0 & 0 & 2k - 6 \end{pmatrix}\) so the system will be inconsistent if \(k \neq 3\) and the system will have infinitely many solutions if \(k = 3\). Note that this system will never have a unique solution since there will always be at least one free variable.

(3) (35 pts) Short answer questions. Justify your answer.

(a) (10 pts) Explain why the set of all continuous functions such that \(f(1) = 0\) forms a (vector) subspace of the continuous functions, but the set of functions such that \(f(0) = 1\) does not.

(b) (10 pts) A matrix \(F\) is called skew-symmetric if \(F^T = -F\). Suppose \(F\) is skew-symmetric, show that \(F^{-1}\) is skew-symmetric.

(c) (8 pts) Are the vectors \(1, 2x - 1, (x - 1)^2, x^2 \in P(2)\) linearly independent? Why or why not?

(d) (7 pts) Suppose \(S = \text{span}\{1, 2x - 1, (x - 1)^2, x^2\}\). Find a basis for \(S\). What is the dimension of \(S\)? Justify your answer.

Solution:

(a) The set \(S = \{f \in C^0|f(1) = 0\}\) forms a subspace since it satisfies closure properties, i.e. if \(f, g \in S\) then for any \(c \in \mathbb{R}\) we have \((cf + g)(1) = c \cdot f(1) + g(1) = c \cdot 0 + 0 = 0\) and so \(cf + g \in S\) while the set \(T = \{f \in C^0|f(0) = 1\}\) does not satisfy closure properties, i.e. note that the zero vector \(O(x) = 0\) is not in the set \(T\) and therefore \(T\) is NOT a vector subspace.

(b) Note \((F^{-1})^T = (F^T)^{-1} = (-F)^{-1} = -F^{-1}\), so \(F\) is skew-symmetric.

(c) No, they are not linearly independent by a dimension argument, note that \(\dim(P(2)) = 3\) and 4 vectors in a space of dimension 3 cannot be linearly independent otherwise that would contradict the dimension of the space.

(d) The dimension of \(S\) is 3 since \((x - 1)^2 = x^2 - 2x + 1 = x^2 - (2x - 1)\) and so 
\[S = \text{span}\{1, 2x - 1, (x - 1)^2, x^2\} = \text{span}\{1, 2x - 1, x^2\}\]
and the set \(\{1, 2x - 1, x^2\}\) is a linearly independent set and therefore forms a basis for \(S\).