1. Let \((X, d)\) be a metric space. Let \(F\) be a closed subset of \(X\) and let \(K\) be a compact subset of \(X\). Prove that
\[
F \cap K \neq \emptyset \iff \inf\{d(x, y) : x \in F, y \in K\} = 0.
\]

2. Let \(\mathbb{X}\) be the set of all sequences in \(\mathbb{R}\) that converge to 0. For \((x_n), (y_n) \in \mathbb{X}\), define the metric
\[
d((x_n), (y_n)) = \sup_n |x_n - y_n|.
\]
Verify that this is a metric and show that the metric space \((\mathbb{X}, d)\) is separable.

3. Let \(A\) be any non-empty set and let \((\mathbb{Y}, d_\mathbb{Y})\) be a complete metric space. Let
\[
\mathbb{X} = \{f : A \to \mathbb{Y} \mid f \text{ is bounded}\}
\]
Define the metric
\[
d_\mathbb{X}(f, g) = \sup_{a \in A} d_\mathbb{Y}(f(a), g(a)).
\]
Show that \((\mathbb{X}, d_\mathbb{X})\) is complete. (You do not need to verify that \(d_\mathbb{X}\) is a metric.)

4. (H & N 1.27) Suppose that \((x_n)\) is a sequence in a compact metric space with the property that every convergent subsequence has the same limit \(x\). Prove that \(x_n \to x\).

5. (H & N 2.3) Suppose that \(f : G \to \mathbb{R}\) is a uniformly continuous function defined on an open subset \(G\) of a metric space \(X\). Prove that \(f\) has a unique extension to a continuous function \(\overline{f} : \overline{G} \to \mathbb{R}\) defined on the closure of \(G\). Show that such an extension need not exist if \(f\) is continuous but not uniformly continuous on \(G\).

6. (H & N 2.5) Consider the space of continuously differentiable functions,
\[
C^1([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f, f' \text{ are continuous}\},
\]
with the \(C^1\)-norm,
\[
||f|| = \sup_{a \leq x \leq b} |f(x)| + \sup_{a \leq x \leq b} |f'(x)|.
\]
Prove that \(C^1([a, b])\) is a Banach space.