APPM/MATH 4/5520
Problem Set Five (Due Wednesday, September 30th)

1. Suppose that \(X_1, X_2, \ldots, X_n\) is a random sample from the \(\Gamma(\alpha, \beta)\) distribution where \(\alpha\) is fixed and known. Find an unbiased estimator of \(\beta\).

2. Let \(X_1, X_2, \ldots, X_n\) be a random sample from the logistic distribution with parameters \(\alpha\) and \(\beta\). (Look it up on the table of distributions!) Find an unbiased estimator of \(\beta^2\).

3. Let \(X \sim \text{exp}(\text{rate} = \lambda)\). Find the exact value of \(P(|X - \mu_X| \geq k\sigma_X)\) for any \(k > 1\). Compare this to the bound you get from Chebyshev’s inequality. Does Chebyshev’s inequality give a useful bound?

4. Suppose that \(X_1, X_2, \ldots, X_n\) is a random sample from the \(\text{unif}(0, \theta)\) distribution.
   (a) Find an unbiased estimator of \(\theta\) based on \(X(n)\), the maximum value of the sample. Call it \(\hat{\theta}_1\).
   (b) Show that \(\hat{\theta}_1\) is a consistent estimator of \(\theta\).
   (c) Show, even though it’s biased for \(\theta\), that \(X(n) \xrightarrow{P} \theta\).

5. Suppose that \(X_1, x_2, \ldots, X_n \sim \text{unif}(0, 1)\). Investigate the convergence in distribution of \(Y_n = 3n(1 - X(n))\) where \(X(n) = \max(X_1, X_2, \ldots, X_n)\).

6. Let \(X(n) = \max(X_1, X_2, \ldots, X_n)\) where \(X_1, X_2, \ldots, X_n\) be a random sample from any continuous distribution that has cdf \(F(x)\) and pdf \(f(x)\). Define \(Z_n = n[1 - F(X(n))]\). Find the limiting distribution (convergence in distribution) of \(Z_n\).

7. Required for 5520 Students Only: Suppose that \(\{X_n\}\) is a sequence of random variables and that \(X\) is another random variable such that
   \[
   \lim_{n \to \infty} \mathbb{E}[(X_n - X)^2] = 0.
   \]
   Show that \(X_n \xrightarrow{P} X\).

8. Required for 5520 Students Only: Suppose that \(\{X_n\}\) is a sequence of random variables with \(X_n \xrightarrow{P} X\) and that \(g\) is a continuous function. Show that \(g(X_n) \xrightarrow{P} g(X)\).