Theorem: A subset of a metric space is compact if and only if it is sequentially compact.

Proof:

⇒ Suppose that \((X, d)\) is a compact metric space. Further, suppose that it is not sequentially compact.

- If \(X\) is not sequentially compact, there exists a sequence \((x_n)\) in \(X\) that has no convergent subsequence. Since there is no convergent subsequence, \((x_n)\) must contain an infinite number of distinct points. (If there were only a finite number of distinct points, the sequence would eventually become constant and would therefore be convergent and thus all subsequences would be convergent!)

- Let \(x \in X\). If, for every \(\varepsilon > 0\), the ball \(B_\varepsilon(x)\) contains a point in the sequence \((x_n)\) that is distinct from \(x\), the \(x\) will be the limit of a subsequence since we would be able to choose points from \((x_n)\) from shrinking balls around \(x\). So, there is a \(\varepsilon_x > 0\) such that \(B_{\varepsilon_x}(x)\) contains no points from \((x_n)\), except possibly \(x\) itself.

- The collection of open balls \(\{B_{\varepsilon_x}(x) : x \in X\}\) is an open cover of \(X\).

- The union of every finite number of these balls contains at most \(n\) terms in the sequence. Because there are an infinite number of distinct terms in the sequence, no finite subcollection of these balls will cover \(X\) since no finite subcollection will even cover the terms of the sequence \((x_n)\) in \(X\).

- So, we have found an open cover of \(X\) that has no finite subcover. This contradicts that \(X\) is compact. Therefore, \(X\) must be sequentially compact.

⇐ Now suppose that \((X, d)\) is sequentially compact. Let \(\{G_\alpha\}\) be an arbitrary open cover of \(X\).

- From the Lemma at the beginning of this solutions, \(X\) is separable which means that \(X\) contains a countable dense subset \(A\).

- Let \(B\) be the collection of open balls with rational radius and center in \(A\). Since \(A\) is countable and the rationals are countable, \(B\) is countable.

- Let \(C\) be the subcollection of balls in \(B\) that are contained in at least one of the open sets in the cover \(\{G_\alpha\}\). Since \(C\) is a subset of \(B\) and \(B\) is countable, \(C\) is countable.

- For every \(x \in X\) there is a \(G_\alpha\) such that \(x \in G_\alpha\). Since \(G_\alpha\) is open, there exists an \(\varepsilon > 0\) such that \(B_\varepsilon(x) \subseteq G_\alpha\).

- Since \(A\) is dense in \(X\), there exists a point \(y \in A\) that is within \(\varepsilon/3\) of \(x\). Note then that \(x \in B_{\varepsilon/3}(y)\) and that \(B_{2\varepsilon/3}(y) \subseteq G_\alpha\).
• Take $q \in \mathbb{Q}$ such that $\varepsilon/3 < q < 2\varepsilon/3$. Then $x \in B_q(y) \subseteq B_{2\varepsilon/3}(y) \subseteq G_\alpha$. Since $B_q(y)$ has rational radius and center in $A$ it is a ball in $B$. Furthermore, since it is a ball in $B$ that is contained in a $G_\alpha$, it is in the collection $C$.

• Thus, every $x \in X$ belongs to a ball in $C$. So, $C$ is a countable open cover of $X$!

• Every ball $B \in C$ is in at least one set $G_\alpha$ in $\{G_\alpha\}$. Pick an index $\alpha_B$ such that $B \subseteq G_{\alpha_B}$. Since $C$ is countable and covers $X$ and since $\{G_{\alpha_B}|B \in C\}$ covers $C$, $\{G_{\alpha_B}|B \in C\}$ countable subcover (of the open cover $\{G_\alpha\}$) of $X$.

• We wanted to show that an open cover of a sequentially compact space has a finite subcover. So far, we have shown that it has a countable subcover. We will now show that a countable open cover of a sequentially compact space has a finite subcover. Ignore all of the previous notation and assume that $\{G_n\}$ is a countable open cover of $X$. Assume that there is no finite subcover. We are going to construct a sequence in $X$ that has no convergent subsequence, thereby contradicting that $X$ is sequentially compact.

• Since $\{G_n\}$ has no finite subcover, $\cup_{k=1}^n G_k$ does not contain $X$ for any $n$.

Construction of the sequence:

- Choose $x_1 \in X$. Since $\{G_n\}$ covers $X$, there exists an $n_1$ such that $x_1 \in G_{n_1}$.

- Choose $x_2 \in X$ such that $x_2 \notin \cup_{n=1}^{n_1} G_n$. We can do this because we have assumed that $X$ can not be covered by a finite subset of $\{G_n\}$. Since $\{G_n\}$ covers $X$, there exists an $n_2$ such that $x_2 \in G_{n_2}$.

- Choose $x_3 \in X$ such that $x_3 \notin \cup_{n=1}^{n_3} G_n$. Choose $n_3$ so that $x_3 \in G_{n_3}$.

- Et cetera! Note that $x_k \in G_{n_k}$ and $x_k \notin \cup_{n=1}^{n_k-1} G_n$.

So, $G_{n_k}$ is not equal to $G_n$ for any $n = 1, 2, \ldots, n_{k-1}$, and the sequence $(n_k)$ is strictly increasing.

• Since $X$ is sequentially compact, $(x_n)$ must have a subsequence that converges to a point $x \in X$. Since $\{G_n\}$ covers $X$, $x \in G_n$ for some $n$.

• However, by construction of our sequence, there exists an integer $K_n$ such that $x_k \notin G_n$ for all $k \geq K_n$.

• $x \in G_n$ yet the sequence $(x_n)$, and hence any subsequence of $(x_n)$ can not be in $G_n$ after some point. This contradicts the statement that $(x_n)$ must have a subsequence converging to $x$ and the sequential compactness of $X$.

• Therefore, the open cover $\{G_n\}$ must have a finite subcover and $X$ is compact.