**Theorem:** Let $X$ be a metric space. A set $F \subseteq X$ is closed in $X$ if and only if every convergent sequence $(x_n)$ with $x_n \in F$ for all $n$ converges to an element in $F$. (We say that “$F$ contains all of its limit points”.)

**Proof:**

$(\Rightarrow)$ Suppose that $F$ is closed.

- Let $(x_n)$ be a convergent sequence with $x_n \in F \ \forall \ n$ and let $x = \lim_{n \to \infty} x_n$.
- Suppose that $x \notin F$. Then $x \in F^c$.
- $F$ open $\Rightarrow F^c$ open $\Rightarrow \exists \ \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq F$.
- Since $x_n \to x$, we can find an $N \in \mathbb{N}$ such that $d(x_n, x), \varepsilon \ \forall \ n \geq N$.
- This implies that $x_n \in B_\varepsilon(x) \subseteq F^c$ for all $n \geq N$.
- But this contradicts the assumption that $x_n \in F \ \forall \ n$. Therefore, $F$ must be open.

Proof:

$(\Leftarrow)$ Suppose now that $F$ contains all of its limit points.

- Suppose that $F$ is not closed.
- Then $F^c$ is not open.
- So, there exists an element $x \in F^c$ such that every ball of any radius centered at $x$ contains at least one point in $(F^c)^c = F$.
- For $n = 1, 2, \ldots$, choose a point $x_n \in B_{1/n}(x) \cap F$.
- Then $(x_n)$ is a sequence in $F$ with a limit $x$ that is not in $F$.
- This contradicts that $F$ contains all of its limit points so $F$ must be closed.