APP 5440: Applied Analysis I
Problem Set Four (Due Friday, September 25th)

1. A metric space \((X, d)\) is said to be an **ultrametric space** if

\[ d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad \forall \ x, y, z \in X. \]

Prove that in an ultrametric space, every open ball \(B_r(x)\) is also closed.

2. Recall our proof that every metric space has a completion. Let \((X, d)\) be a metric space, let \(C(X)\) be the set of all Cauchy sequences in \(X\), and let \(\sim\) be the equivalence relation on \(C(X)\) defined as

\[(x_n) \sim (y_n) \iff \lim_{n \to \infty} d(x_n, y_n) = 0.\]

Suppose that \((x_n)\) and \((y_n)\) are in the same equivalence class in \(C(X)\) and that \((x_{n_k})\) and \((y_{m_k})\) are subsequences of \((x_n)\) and \((y_n)\), respectively. Show that \((x_{n_k}) \sim (y_{m_k})\).

3. A metric space \((X, d)\) is **totally bounded** if, for every \(\varepsilon > 0\), there exist a finite number of open balls of radius \(\varepsilon\) that covers \(X\)

A **subset** of a metric space is totally bounded if, for every \(\varepsilon > 0\), there exist a finite number of open balls of radius \(\varepsilon\) that covers the subset.

Prove that a subset of a metric space is sequentially compact if and only if it is complete and totally bounded.

4. (H & N 1.15) Prove that every compact subset of a metric space is closed and bounded. Prove that a closed subset of a compact space is compact.

5. (H & N 1.23) Let \(X\) be a metric space. A function \(f : X \to \mathbb{R}\) is said to be **lower semicontinuous** on \(X\) if for all \(x \in X\) and every sequence \((x_n)\) that converges to \(x\), we have

\[ \liminf_{n \to \infty} f(x_n) \geq f(x). \]

(It is upper semicontinuous if \(\limsup_{n \to \infty} f(x_n) \leq x\).)

Suppose that \(f : X \to \mathbb{R}\) is lower semicontinuous and \(M\) is a real number. Define \(f_M : X \to \mathbb{R}\) by

\[ f_M(x) = \min(f(x), M). \]

Prove that \(f_M\) is also lower semicontinuous.