Section 1.5

5. (a) \( y' = \frac{1}{t^2 + y^2}, \, y(0) = 0 \)

Here both

\[
 f(t, y) = \frac{1}{t^2 + y^2} \\
 f_y(t, y) = -\frac{2y}{(t^2 + y^2)^2}
\]

are continuous for all \( t \) and \( y \) except at the point \( y = t = 0 \). Hence, there is a unique solution passing through any initial point \( y(t_0) = y_0 \) except \( y(0) = 0 \). In this case \( f \) does not exist, so the IVP does not make sense. The direction field of the equation illustrates these ideas (see figure).

(b) Picard's Theorem gives existence/uniqueness for any rectangle that does not include the origin.

(c) It may be useful to replace the initial condition \( y(0) = 0 \) by \( y(0) = y_0 \) with small but nonzero \( y_0 \).

6. (a) \( y' = \tan y, \, y(0) = \frac{\pi}{2} \)

Here

\[
 f(t, y) = \tan y \\
 f_y = \sec^2 y
\]

are both continuous except at the points

\( y = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \).

Hence, there exists a unique solution passing through \( y(t_0) = y_0 \) except when

\( y = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \).

The IVP problem passing through \( \frac{\pi}{2} \) does not have a solution. It would be useful to look at the direction field to get an idea of the behavior of solutions for nearby initial points. The direction field of the equation shows that where Picard's Theorem does not work the slope has become vertical (see figure).
21. \( y' = \sqrt{y} \), \( y(0) = 0 \), \( t_0 > 0 \)

For \( t < t_0 \), the solution is \( y(t) \equiv 0 \). For \( t > t_0 \), we have \( y = \frac{1}{4}(t-t_0)^2 \).

At \( t = t_0 \) the left-hand derivative of \( y(t) \equiv 0 \) is 0, and the right-hand derivative of \( y(t) = \frac{1}{4}(t-t_0)^2 \) is 0, so they agree.

27. (a) \( y' = y \) has an infinite family of solution of the form \( y = Ce^t \).

(To check: \( y' = (Ce^t)' = Ce^t = y \).)

Note that for any real number \( a \), \( y = e^{-x} \) is a solution for every \( a \in \mathbb{R} \).

(b) Differentiating \( s(t) = \begin{cases} 0 & t < a \\ (t-a) & t \geq a \end{cases} \)

we obtain a continuous derivative

\[ s'(t) = \begin{cases} 0 & t < a \\ 2(t-a) & t \geq a \end{cases} \]

Note that \( s' = 2\sqrt{s} \) for both parts of the curve.

(c) [Some solutions for \( y' = y \).]

[Some solutions for \( s' = 2\sqrt{s} \).]

For (a), with \( y' = y \), solutions \( y = Ce^t \) gradually approach zero as \( t \to -\infty \).

For (b), with \( s' = 2\sqrt{s} \), solutions \( y = \begin{cases} 0 & \text{for } t < a \\ (t-a)^2 & \text{for } t \geq a \end{cases} \) go to zero at \( t \to a \).
Section 2.1

3. Second order, linear, homogeneous, variable coefficients
6. Third order, linear, non-homogeneous, constant coefficient
8. Second order, nonlinear

14. \( L(y) = y'' + 2y \)

Suppose \( y_1, y_2 \) and \( y \) are functions and \( c \) is any constant.

\[
L(y_1 + y_2) = (y_1 + y_2)'' + 2(y_1 + y_2) \\
= (y_1'' + 2y_1) + (y_2'' + 2y_2) \\
= L(y_1) + L(y_2)
\]

\[
L(cy) = (cy)'' + 2(cy) = c(y'' + 2y)
\]

Hence, \( L \) is a linear operator. This problem illustrates the fact that the coefficients of a DE can be functions of \( t \) and the operator will still be linear.

15. \( L(y) = y' - \delta y \)

Suppose \( y_1, y_2 \) and \( y \) are functions of \( t \) and \( c \) is any constant.

\[
L(y_1 + y_2) = (y_1 + y_2)' - \delta (y_1 + y_2) \\
= (y_1' - \delta y_1) + (y_2' - \delta y_2) \\
= L(y_1) + L(y_2)
\]

\[
L(cy) = (cy)' - \delta (cy) = c(y' - \delta y)
\]

Hence, \( L \) is a linear operator. This problem illustrates the fact that a linear operator need not have coefficients that are linear functions of \( t \).

17. \( L(y) = y'' + (1 - y^2) y' + y \)

\[
L(cy) = (cy)'' + \left[1 - (cy)^2\right] y' + (cy) \\
= c \left[y'' + (1 - y^2) y' + y\right] \\
= cL(y)
\]

Hence, \( L(y) \) is not a linear operator.

25. If \( y_1 \) and \( y_2 \) are solutions of \( y'' + p(t)y' + q(t)y = 0 \),
we have

\[
y_1'' + p(t)y_1' + q(t)y_1 = 0 \\
y_2'' + p(t)y_2' + q(t)y_2 = 0.
\]

Multiplying these equations by \( c_1 \) and \( c_2 \) respectively, then adding and using properties of the derivative, we arrive at

\[
(c_1y_1 + c_2y_2)' + p(t)(c_1y_1 + c_2y_2) + q(t)(c_1y_1 + c_2y_2) = 0,
\]

which shows that \( c_1y_1 + c_2y_2 \) is also a solution.
30. \( y'' - y' - 6y = 0 \)

For \( y_1 = e^t \), \( y_1' = 3e^t \), \( y_1'' = 9e^t \)
Substituting: \( (9e^t) - (3e^t) - 6(e^t) = 0 \)

For \( y_2 = e^{-2t} \), \( y_2' = -2e^{-2t} \), \( y_2'' = 4e^{-2t} \)
Substituting: \( (4e^{-2t}) - (-2e^{-2t}) - 6e^{-2t} = 0 \)

For \( y = c_1e^t + c_2e^{-2t} \)
\[ y'' - y' - 6y = \left( c_1 9e^t + c_2 4e^{-2t} \right) - \left( c_1 3e^t - 2e^{-2t} \right) - 6 \left( c_1 e^t + c_2 e^{-2t} \right) = 0. \]

31. \( y'' - 9y = 0 \)

For \( y_1 = \cosh 3t \), \( y_1' = 3 \sinh 3t \), \( y_1'' = 9 \cosh 3t \).
Substituting: \( (9 \cosh 3t) - 9(\cosh 3t) = 0 \).

For \( y_2 = \sinh 3t \), \( y_2' = 3 \cosh 3t \), \( y_2'' = 9 \sinh 3t \).
Substituting: \( (9 \sinh 3t) - 9(\sinh 3t) = 0 \).

For \( y = c_1 \cosh 3t + c_2 \sinh 3t \),
\[ y'' - 9y = \left( c_1 9 \cosh 3t + c_2 9 \sinh 3t \right) - 9 \left( c_1 \cosh 3t + c_2 \sinh 3t \right) = 0. \]
Section 2.2

6. \( y' + 2ty = t \)

In this problem we see that \( y_p(t) = \frac{1}{2} \) is a solution of the nonhomogeneous equation (there are other single solutions, but this is the easiest to find). Hence, to find the general solution we solve the corresponding homogeneous equation, \( y' + 2ty = 0 \),

by separation of variables, getting

\[
\frac{dy}{y} = -2tdt,
\]

which has the general solution \( y = ce^{t^2} \), where \( c \) is any constant.

Adding the solutions of the homogeneous equation to the particular solution \( y_p = \frac{1}{2} \) we get the general solution of the nonhomogeneous equation:

\[
y(t) = ce^{t^2} + \frac{1}{2}.
\]

12. \( y' + \frac{3}{t}y = \sin t \), \( t \neq 0 \)

We multiply each side of the equation by the integrating factor

\[
\mu(t) = e^{\int \frac{3}{t} dt} = e^{3\ln t} = t^3,
\]

giving

\[
t^3 \left( y' + \frac{3}{t}y \right) = \sin t, \text{ or, } \frac{d}{dt} \left( t^3 y \right) = \sin t.
\]

Integrating, we find \( t^3 y = -\cos t + c \).

Solving for \( y \), we get \( y(t) = \frac{c}{t^3} - \frac{1}{t^3} \cos t \).

17. \( y' + 2ty = t^3 \), \( y(1) = 1 \)

We can solve the differential equation using either the Euler-Lagrange method or the integrating factor method to get

\[
y(t) = \frac{1}{2} t^2 - \frac{1}{2} + ce^{-t^2}.
\]

Substituting \( y(1) = 1 \) we find \( ce^{-1} = 1 \) or \( c = e \). Hence, the solution of the IVP is

\[
y(t) = \frac{1}{2} t^2 - \frac{1}{2} + e^{-t^2}.
\]

28. \( y' + 3ty = t^2 \)

Here \( p(t) = 3t^2 \), therefore the integrating factor is \( \mu(t) = e^{\int p(t) dt} = e^{3t^3/3} = e^{t^3} \).

Multiplying each side of the equation \( y' + 3ty = t^2 \) by \( e^{t^3} \) yields

\[
\frac{d}{dt} \left( ye^{t^3} \right) = t^2 e^{t^3}.
\]

Integrating gives \( ye^{t^3} = \frac{1}{3} e^{t^3} + c \). Solving for \( y \) gives \( y(t) = ce^{t^3} + \frac{1}{3} \).
34. (a) We divide by $y^\alpha$ to obtain $y^{-\alpha} y' + p(t) y^{1-\alpha} = q(t)$.

Let $v = y^{1-\alpha}$ so that $v' = (1-\alpha) y^{-\alpha} y'$ and $\frac{v'}{1-\alpha} = y^{-\alpha} y'$.

Substituting into the first equation for $y^{1-\alpha}$ and $y^{-\alpha} y'$, we have
\[ \frac{v'}{1-\alpha} + p(t)v = q(t), \]
which is a linear DE in $v$.

which we can now rewrite into standard form as
\[ v' + (1-\alpha) p(t)v = (1-\alpha)q(t). \]

(b) $\alpha = 3, \ p(t) = -1, \ q(t) = 1$; hence $\frac{dv}{dt} + 2v = -2$, which has the general solution
\[ v(t) = -1 + ce^{-2t}. \]

Because $v = \frac{1}{y^\alpha}$, this yields
\[ y(t) = (1 + ce^{-2t})^{\frac{1}{\alpha}}. \]

Note, too, that $y = 0$ satisfies the given equation.

(c) When $\alpha = 0$, the Bernoulli equation is
\[ \frac{dy}{dt} + p(t)y = q(t), \]
which is the general first-order linear equation we solved by the integrating factor method and the Euler-Lagrange method.

When $\alpha = 1$ the Bernoulli equation is
\[ \frac{dy}{dt} + p(t)y = q(t)y, \quad \text{or,} \quad \frac{dy}{dt} + (p(t) - q(t))y = 0, \]
which can be solved by separation of variables.
38. \((1 - t^2)y'' - ty' - ny^2 = 0\) (Assume \(|t| < 1\))

\(y^{-2}(1 - t^2)y' - ty' = t\)

Let \(v = y^{-1}\), so \(\frac{dv}{dt} = -y^{-2} \frac{dy}{dt}\).

Substituting in the DE gives

\(-\frac{(1 - t^2)}{dt} \frac{dv}{dt} = t\), so that \(\frac{dv}{dt} + \frac{t}{1 - t^2}v = \frac{-t}{1 - t^2}\),

which is linear in \(v\), with integrating factor \(\mu = e^{\int \frac{t}{1 - t^2} dt} = e^{\frac{1}{2} \log(1 - t^2)} = (1 - t^2)^{-1/2}\).

Thus, \((1 - t^2)^{-1/2} \frac{dv}{dt} + r(1 - t^2)^{-3/2} v = r(1 - t^2)^{-3/2}\), and

\((1 - t^2)^{-1/2} v = \int \frac{-t}{(1 - t^2)^{3/2}} dt = \frac{1}{2} \int \frac{dw}{w^{3/2}} \quad \text{(Substitute} \quad w = 1 - t^2)\)

\[= \frac{1}{2} \left( \frac{1}{w^{1/2}} \right) + c \]

\[= \frac{1}{2} \left( \frac{1}{\sqrt{w}} \right) + c \]

\[= \sqrt{w} + c = (1 - t^2)^{3/2} + c\]

Hence \(v = 1 + c(1 - t^2)^{3/2}\) and substituting back for \(v\) gives \(y(t) = \frac{1}{1 + c(1 - t)^{3/2}}\).