1. On the notes “Spline illustrations” on the class web page, the decay rates for cardinal cubic- and quadratic splines on equispaced grids were given as \((2 - \sqrt{3})^k\) and 1 (no decay), respectively. Demonstrate these results theoretically.

**Hint:** No great rigor is required, but give plausible arguments that lead to these specific factors. Useful ingredients for the cubic spline case are (i) the linear system that is used for determining spline interpolants, and (ii) the notes “Linear recursion relations” (also on the class web page). With regard to the quadratic case, there is a very brief (2-line) argument that leads to the desired result.

2. When using \(B\)-splines, one often needs to calculate the value of these basis functions at a certain location \(z\). There turns out to be a very simple algorithm for this. In the cubic case, let the \(B\)-spline nodes be at \(x_1, x_2, x_3, x_4, x_5\):

i. Write down the five node locations in the first column of the table below.

ii. Check in which subinterval \(z\) falls. If it is between \(x_i\) and \(x_{i+1}\), then enter \(B^i_p = 1 / (x_{i+1} - x_i)\) while leaving the other \(B^0\) entries zero.

iii. Fill in the rest of the \(B\)-entries recursively (somewhat like the Newton divided difference table):

\[
B^i_p(z) = \frac{(z - x_p)B^{i-1}_p(z) + (x_{p+k+1} - z)B^{i-1}_{p+k}(z)}{x_{p+k+1} - x_p}.
\]

Graphically, in the cubic case and assuming the \(z\)-value fell in the second subinterval, the chart will look like

\[
x_1 = \ldots \quad B^5_1 = 0
\]
\[
x_2 = \ldots \quad B^4_1 = \ldots
\]
\[
x_3 = \ldots \quad B^3_1 = \ldots \quad B^3_1 = \ldots
\]
\[
x_4 = \ldots \quad B^2_1 = \ldots \quad B^2_2 = \ldots
\]
\[
x_5 = \ldots \quad B^1_1 = \ldots \quad B^1_1 = \ldots
\]

The value of the \(B\)-spline at \(z\) will be the rightmost entry in this table (however, with a normalization that may not match the standard custom of integrating to one).

By means of this approach, produce a graph of the cubic \(B\)-spline based on the five nodes \(x = [-2 \ -1 \ 0 \ 3 \ 6]\). On the curve, mark the node locations.

**Notes:** (Not needed for solving the homework problem)

i. For \(B\)-splines of higher orders, one simply includes more initial \(x\)-entries, and run the table further to the right.

ii. For each \(z\)-value, several \(B\)-splines are non-zero. Their values at this location \(z\) can be obtained from a single table as above by again extending it to more \(x\)-values (but now not needing to run it out further to the right).
3. The cubic $B$-spline is the cubic spline that transitions away from and then back to identically zero in the shortest distance. Similarly, the figure to the right shows the cubic spline that transitions the fastest from identically one to identically zero on a unit-paced grid.

Determine the exact $y$-values at the two internal nodes of this spline.

4. a. Show that the cubic $B$-spline $B(x)$, defined by the equispaced nodes $x = [-2 \ -1 \ 0 \ 1 \ 2]$, can be expressed in closed form as

$$B(x) = \frac{1}{12} (|x+2|^3 - 4 |x+1|^3 + 6 |x|^3 - 4 |x-1|^3 + |x-2|^3)$$  \hspace{1cm} (1)

b. If we define the Fourier transform of a function $f(x)$ as $\mathcal{F}(f(x)) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-i\omega x} dx$, show (for ex. with use of Mathematica) that

$$\mathcal{F}(B(x)) = \frac{1}{2\pi} \left( \frac{2\sin \frac{\omega}{2}}{\omega} \right)^4$$  \hspace{1cm} (2)

c. Show that this cubic $B$-spline satisfies the identity

$$B(x) = \frac{1}{8} \left( B(2x - 2) + 4B(2x - 1) + 6B(2x) + 4B(2x + 1) + B(2x + 2) \right)$$  \hspace{1cm} (3)

**Hint:** This identity can be proved in various different ways:

(i) Use (2) and show that both sides of (3) have the same Fourier transform.

(ii) Substitute (1) into (3), and use Mathematica to simplify the difference between the two sides.

(iii) Let $B$ in the right hand side (RHS) be the $B$-spline, and then show that the RHS itself becomes a cubic spline. Next show that the RHS shares sufficiently many features with the LHS that, by a uniqueness result for cubic splines, the two sides in fact have to be identically the same.

**Comment:** Equation (3) is an example of a ‘functional equation’, relating a function to itself. In the case of (3), the relation defines $B(x)$ everywhere if one just knows that $B(x)$ is zero at all integer locations apart from $B(\pm 1) = 1/6, B(0) = 2/3$. With this information, equation (3) first gives $B(x)$ at all half-integer locations, then at all quarter-integer locations, etc.