11.4

19. \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad f_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \ f_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \) and

\[ f_z(x, y, z) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \] so

\[ f_x(3, 2, 6) = \frac{3}{7}, \ f_y(3, 2, 6) = \frac{2}{7}, \ f_z(3, 2, 6) = \frac{6}{7}. \] Then the linear approximation of \( f \) at \((3, 2, 6)\) is given by

\[ f(x, y, z) \approx f(3, 2, 6) + f_x(3, 2, 6)(x - 3) + f_y(3, 2, 6)(y - 2) + f_z(3, 2, 6)(z - 6) \]

\[ = 7 + \frac{3}{7}(x - 3) + \frac{2}{7}(y - 2) + \frac{6}{7}(z - 6) = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z \]

Thus \( \sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914. \)

28. Let \( V \) be the volume. Then

\[ V = \pi r^2 h \quad \text{and} \quad \Delta V \approx dV = 2\pi rh \, dr + \pi r^2 \, dh \]

is an estimate of the amount of metal. With \( dr = 0.05 \) and \( dh = 0.2 \) we get

\[ dV = 2\pi(2)(10)(0.05) + \pi(2)^2(0.2) = 2.80\pi \approx 8.8 \text{ cm}^3. \]

30. Here \( dV = \Delta V = 0.3, \ dT = \Delta T = -5, \ P = 8.31 \frac{T}{V}, \) so

\[ dP = \left( \frac{8.31}{V} \right) dT - \frac{8.31 \cdot T}{V^2} \ dV = 8.31 \left[ -\frac{5}{12} + \frac{310}{144} \cdot \frac{3}{10} \right] \approx -8.83. \]

Thus the pressure will drop by about 8.83 kPa.

31. The errors in measurement are at most 2%, so

\[ \left| \frac{\Delta w}{w} \right| \leq 0.02 \quad \text{and} \quad \left| \frac{\Delta h}{h} \right| \leq 0.02. \]

The relative error in the calculated surface area is

\[ \frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{0.1091(0.425u^{0.425} - 1)h^{0.725} dw + 0.1091u^{0.425}h^{0.725} dh}{0.1091u^{0.425}h^{0.725}} = 0.425 \frac{dw}{w} + 0.725 \frac{dh}{h}. \]

To estimate the maximum relative error, we use

\[ \frac{dw}{w} = \left| \frac{\Delta w}{w} \right| = 0.02 \quad \text{and} \quad \frac{dh}{h} = \left| \frac{\Delta h}{h} \right| = 0.02 \implies \]

\[ \frac{dS}{S} = 0.425(0.02) + 0.725(0.02) = 0.023. \]

Thus the maximum percentage error is approximately 2.3%.

34. \( r_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle \quad \Rightarrow \quad r_1'(t) = \langle 3, -2t, -4 + 2t \rangle, \quad r_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle \quad \Rightarrow \]

\[ r_2'(u) = \langle 2u, 6u^2, 2 \rangle. \] Both curves pass through \( P \) since \( r_1(0) = r_2(1) = \langle 2, 1, 3 \rangle, \) so the tangent vectors \( r_1'(0) = \langle 3, 0, -4 \rangle \) and \( r_2'(1) = \langle 2, 6, 2 \rangle \) are both parallel to the tangent plane to \( S \) at \( P. \) A normal vector for the tangent plane is

\[ r_1'(0) \times r_2'(1) = \langle 3, 0, -4 \rangle \times \langle 2, 6, 2 \rangle = \langle 24, -14, 18 \rangle, \] so an equation of the tangent plane is

\[ 24(x - 2) - 14(y - 1) + 18(z - 3) = 0 \quad \text{or} \quad 12x - 7y + 9z = 44. \]
11.5

20. \( P = \sqrt{u^2 + v^2 + w^2} = (u^2 + v^2 + w^2)^{1/2}, \; u = x e^y, \; v = y e^x, \; w = e^{xy} \Rightarrow \)

\[
\frac{\partial P}{\partial x} = \frac{\partial P}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial P}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial P}{\partial w} \frac{\partial w}{\partial x} = 1/2 \left( u^2 + v^2 + w^2 \right)^{-1/2} (u)(e^x) + 1/2 \left( u^2 + v^2 + w^2 \right)^{-1/2} (v)(e^y) + 1/2 \left( u^2 + v^2 + w^2 \right)^{-1/2} (w)(ye^x) = \frac{ue^y + vye^x + wye^{xy}}{\sqrt{u^2 + v^2 + w^2}},
\]

\[
\frac{\partial P}{\partial y} = \frac{\partial P}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial P}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial P}{\partial w} \frac{\partial w}{\partial y} = \frac{u}{\sqrt{u^2 + v^2 + w^2}} (xe^y) + \frac{v}{\sqrt{u^2 + v^2 + w^2}} (ye^x) + \frac{w}{\sqrt{u^2 + v^2 + w^2}} (xe^{xy}) = \frac{uje^y + vxe^x + wxe^{xy}}{\sqrt{u^2 + v^2 + w^2}}.
\]

When \( x = 0 \) and \( y = 2 \) we have \( u = 0 \), \( v = 2 \), and \( w = 1 \), so \( \frac{\partial P}{\partial x} = \frac{0 + 4 + 2}{\sqrt{5}} = \frac{6}{\sqrt{5}} \) and \( \frac{\partial P}{\partial y} = \frac{0 + 2 + 0}{\sqrt{5}} = \frac{2}{\sqrt{5}} \).

32. \( V = \pi r^2 h/3 \) so \( \frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = \frac{2\pi rh}{3}1.8 + \frac{\pi r^2}{3}(-2.5) = 20,160\pi - 12,000\pi = 8160\pi \text{ in}^3/\text{s}.

34. \( I = \frac{V}{R} \Rightarrow \)

\[
\frac{dI}{dt} = \frac{\partial I}{\partial V} \frac{dV}{dt} + \frac{\partial I}{\partial R} \frac{dR}{dt} = \frac{1}{R} \frac{dV}{dt} - \frac{1}{R^2} \frac{dR}{dt} = \frac{1}{400} (-0.01) - \frac{0.08}{400} (0.03) = -0.000031 \text{ A/s}
\]

47. \( F(x, y, z) = 0 \) is assumed to define \( z \) as a function of \( x \) and \( y \), that is, \( z = f(x, y) \). So by (7), \( \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \) since \( F_z \neq 0 \).

Similarly, it is assumed that \( F(x, y, z) = 0 \) defines \( x \) as a function of \( y \) and \( z \), that is \( x = h(x, z) \). Then \( F(h(y, z), y, z) = 0 \)

and by the Chain Rule, \( F_x \frac{\partial x}{\partial y} + F_y \frac{\partial y}{\partial y} + F_z \frac{\partial z}{\partial y} = 0 \). But \( \frac{\partial x}{\partial y} = 0 \) and \( \frac{\partial y}{\partial y} = 1 \), so \( F_x \frac{\partial x}{\partial y} + F_y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} \).

A similar calculation shows that \( \frac{\partial y}{\partial z} = -\frac{F_x}{F_y} \). Thus \( \frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = \left( \frac{F_z}{F_x} \right) \left( \frac{F_x}{F_y} \right) \left( \frac{F_z}{F_y} \right) = -1 \).

11.6
48. Given a function defined implicitly by \( F(x, y) = 0 \), where \( F \) is differentiable and \( F_y \neq 0 \), we know that \( \frac{dy}{dx} = -\frac{F_x}{F_y} \). Let \( G(x, y) = \frac{F_x}{F_y} \) so \( \frac{dy}{dx} = G(x, y) \). Differentiating both sides with respect to \( x \) and using the Chain Rule gives

\[
\frac{d^2y}{dx^2} = \frac{\partial G}{\partial x} \frac{dx}{dx} + \frac{\partial G}{\partial y} \frac{dy}{dx} = \frac{\partial}{\partial x} \left( \frac{F_x}{F_y} \right) - \frac{F_x F_{xx} - F_y F_{xy}}{F_y^2}, \quad \frac{\partial G}{\partial y} \frac{dy}{dx} = -\frac{F_x}{F_y}.
\]

Thus

\[
\frac{d^2y}{dx^2} = \left( -\frac{F_x F_{xx} - F_y F_{xy}}{F_y^2} \right) + \left( -\frac{F_x F_{xy} - F_y F_{yy}}{F_y^2} \right) \left( -\frac{F_x}{F_y} \right)
\]

\[
= -\frac{F_{xx} F_y^2 - F_{xy} F_x F_y - F_{xy} F_x F_y + F_{yy} F_x^2}{F_y^3}
\]

But \( F \) has continuous second derivatives, so by Clairaut’s Theorem, \( F_{yy} = F_{xy} \) and we have

\[
\frac{d^2y}{dx^2} = -\frac{F_{xx} F_y^2 - 2F_{xy} F_x F_y + F_{yy} F_x^2}{F_y^3}
\]
as desired.

22. The fisherman is traveling in the direction \((-80, -60)\). A unit vector in this direction is \( \mathbf{u} = \frac{1}{100} (-80, -60) = \left(-\frac{4}{5}, -\frac{3}{5}\right) \), and if the depth of the lake is given by \( f(x, y) = 200 + 0.02x^2 - 0.001y^2 \), then \( \nabla f(x, y) = (0.04x, -0.003y^2) \).

\[
D_u f(80, 60) = \nabla f(80, 60) \cdot \mathbf{u} = (3.2, -10.8) \cdot \left(-\frac{4}{5}, -\frac{3}{5}\right) = 3.92.
\]
Since \( D_u f(80, 60) \) is positive, the depth of the lake is increasing near \((80, 60)\) in the direction toward the buoy.

23. \( T = \frac{k}{\sqrt{x^2 + y^2 + z^2}} \) and \( 120 = T(1, 2, 2) = \frac{k}{3} \) so \( k = 360 \).

(a) \( \mathbf{u} = \frac{(1, -1, 1)}{\sqrt{3}} \)

\[
D_u T(1, 2, 2) = \nabla T(1, 2, 2) \cdot \mathbf{u} = \left[ -360(x^2 + y^2 + z^2)^{-3/2}(x, y, z) \right]_{(1, 2, 2)} \cdot \mathbf{u} = -\frac{40}{3}(1, 2, 2) \cdot \frac{1}{\sqrt{3}} (1, -1, 1) = -\frac{40}{3\sqrt{3}}
\]

(b) From (a), \( \nabla T = -360(x^2 + y^2 + z^2)^{-3/2}(x, y, z) \), and since \((x, y, z)\) is the position vector of the point \((x, y, z)\), the vector \(-\langle x, y, z \rangle\), and thus \( \nabla T \), always points toward the origin.

26. \( z = f(x, y) = 1000 - 0.005x^2 - 0.01y^2 \Rightarrow \nabla f(x, y) = (-0.01x, -0.02y) \) and \( \nabla f(60, 40) = (-0.6, -0.8) \).

(a) Due south is in the direction of the unit vector \( \mathbf{u} = -\mathbf{j} \)

\[
D_u f(60, 40) = \nabla f(60, 40) \cdot (0, -1) = (-0.6, -0.8) \cdot (0, -1) = 0.8.
\]
Thus, if you walk due south from \((60, 40, 966)\) you will ascend at a rate of 0.8 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector \( \mathbf{u} = \frac{1}{\sqrt{2}} (-1, -1) \)

\[
D_u f(60, 40) = \nabla f(60, 40) \cdot \frac{1}{\sqrt{2}} (-1, -1) = (-0.6, -0.8) \cdot \frac{1}{\sqrt{2}} (-1, -1) = -\frac{0.4}{\sqrt{2}} \approx -0.14.
\]
Thus, if you walk northwest from \((60, 40, 966)\) you will descend at a rate of approximately 0.14 vertical meters per horizontal meter.

(c) \( \nabla f(60, 40) = (-0.6, -0.8) \) is the direction of largest slope with a rate of ascent given by

\[
|\nabla f(60, 40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1.
\]
The angle above the horizontal in which the path begins is given by

\[
\tan \theta = 1 \Rightarrow \theta = 45^\circ.
\]
40. \( g(x, y) = x^2 + y^2 - 4x \) \( \Rightarrow \) \( \nabla g(x, y) = \langle 2x - 4, 2y \rangle \),

\( \nabla g(1, 2) = \langle -2, 4 \rangle \). \( \nabla g(1, 2) \) is perpendicular to the tangent line, so

the tangent line has equation \( \nabla g(1, 2) \cdot (x - 1, y - 2) = 0 \) \( \Rightarrow \)

\( \langle -2, 4 \rangle \cdot (x - 1, y - 2) = 0 \) \( \Rightarrow \) \(-2(x - 1) + 4(y - 2) = 0 \) \( \iff \)

\(-2x + 4y = 6 \) or equivalently \(-x + 2y = 3\).

42. Let \( F(x, y, z) = x^2 + z^2 - y \); then the paraboloid \( y = x^2 + z^2 \) is a level surface of \( F \). \( \nabla F(x, y, z) = \langle 2x, -1, 2z \rangle \) is a normal vector to the surface at \((x, y, z)\) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane \( x + 2y + 3z = 1 \) when the normal vectors of the planes are parallel, so we need a point \((x_0, y_0, z_0)\) on the paraboloid where \( \langle 2x_0, -1, 2z_0 \rangle = k \langle 1, 2, 3 \rangle \). Comparing \( y \)-components we have \( k = -\frac{1}{2} \), so

\( \langle 2x_0, -1, 2z_0 \rangle = \langle -\frac{1}{2}, -1, -\frac{3}{2} \rangle \) and \( 2x_0 = -\frac{1}{2} \) \( \Rightarrow \) \( x_0 = -\frac{1}{4}, 2z_0 = -\frac{3}{2} \) \( \Rightarrow \) \( z_0 = -\frac{3}{4} \). Then

\( y_0 = x_0^2 + z_0^2 = \left(-\frac{1}{4}\right)^2 + \left(-\frac{3}{4}\right)^2 = \frac{5}{8} \) and the point is \( \left(-\frac{1}{4}, \frac{5}{8}, -\frac{3}{4}\right) \).

44. First note that the point \((1, 1, 2)\) is on both surfaces. The ellipsoid is a level surface of \( F(x, y, z) = 3x^2 + 2y^2 + z^2 \) and \( \nabla F(x, y, z) = \langle 6x, 4y, 2z \rangle \). A normal vector to the surface at \((1, 1, 2)\) is \( \nabla F(1, 1, 2) = \langle 6, 4, 4 \rangle \) and an equation of the tangent plane there is \( 6(x - 1) + 4(y - 1) + 4(z - 2) = 0 \) or \( 3x + 4y + 4z = 18 \) or \( 3x + 2y + 2z = 9 \). The sphere is a level surface of \( G(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 \) and \( \nabla G(x, y, z) = \langle 2x - 8, 2y - 6, 2z - 8 \rangle \). A normal vector to the sphere at \((1, 1, 2)\) is \( \nabla G(1, 1, 2) = \langle -6, -4, -4 \rangle \) and the tangent plane there is \(-6(x - 1) - 4(y - 1) - 4(z - 2) = 0 \) or \( 3x + 2y + 2z = 9 \). Since these tangent planes are identical, the surfaces are tangent to each other at the point \((1, 1, 2)\).
(a) We want to show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point $P$ where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{and} \quad P.$$

This statement is an “if and only if”, which means we have to prove two statements: $F$ and $G$ orthogonal implies $F_x G_x + F_y G_y + F_z G_z = 0$, and $F_x G_x + F_y G_y + F_z G_z = 0$ implies $F$ and $G$ are orthogonal.

For the first statement, assume that $F$ and $G$ are orthogonal at $P$, where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$. By definition, this means that the normals to the surfaces described by $F$ and $G$ are orthogonal. This means that $\nabla F \perp \nabla G$. Therefore

$$\nabla F \cdot \nabla G = F_x G_x + F_y G_y + F_z G_z = 0.$$

For the second statement, assume that $F_x G_x + F_y G_y + F_z G_z = 0$ at $P$. We can factor this as

$$0 = \langle F_x, F_y, F_z \rangle \cdot \langle G_x, G_y, G_z \rangle = \nabla F \cdot \nabla G.$$

This means $\nabla F \perp \nabla G$ at $P$. Since $\nabla F$ and $\nabla G$ are normal to $F$ and $G$, respectively, we know that $F$ and $G$ are orthogonal at $P$.

(b) Define

$$F(x, y, z) = x^2 + y^2 - z^2 = 0$$
$$G(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0.$$

We can now directly compute $\nabla F \cdot \nabla G$:

$$\nabla F \cdot \nabla G = \langle 2x, 2y, -2z \rangle \cdot \langle 2x, 2y, 2z \rangle = 4 \left( x^2 + y^2 - z^2 \right).$$

Now notice that $x^2 + y^2 - z^2 = F(x, y, z) = 0$ for any $(x, y, z)$ on the surface $F$. On the intersection of $F$ and $G$, we must have $F(x, y, z) = 0$ and $G(x, y, z) = 0$. Therefore, along the intersection,

$$\nabla F \cdot \nabla G = 4F(x, y, z) = 0,$$

so the surfaces are orthogonal at every point of intersection.

(c) Note that $F$ defines a circular cone and $G$ defines a sphere of radius $r$. The vector $\nabla G$ at a point $P$ is always parallel to the vector from the center of the sphere to the point $P$. The sides of the cone $F$ are always along the radius of the sphere (parallel to $\nabla G$), so the surfaces must be orthogonal at their intersection.
Let \( \hat{u} = \langle u_1, u_2 \rangle \) and \( \hat{v} = \langle v_1, v_2 \rangle \) be two nonparallel unit vectors. We are given \( \frac{\partial f}{\partial s} = \nabla f \big|_P \cdot \hat{u} = A \) in the \( \hat{u} \) direction, and \( \frac{\partial f}{\partial s} = \nabla f \big|_P \cdot \hat{v} = B \) in the \( \hat{v} \) direction. Expanding the dot products into components gives

\[
\begin{align*}
&f_x u_1 + f_y u_2 = A \\
&f_x v_1 + f_y v_2 = B,
\end{align*}
\]

where \( f_x = f_x \big|_P = \frac{\partial f}{\partial x} \big|_P \) and \( f_y = f_y \big|_P = \frac{\partial f}{\partial y} \big|_P \). We now have two equations in the two unknowns, \( f_x \) and \( f_y \); if we can solve this system for \( f_x \) and \( f_y \), we can recover the gradient \( \nabla f \big|_P \).

Using standard algebraic techniques (e.g. adding \( -\frac{v_1}{u_1} \) times the first equation to the second) we find

\[
\begin{align*}
&f_x = \frac{Av_2 - Bu_2}{u_1 v_2 - u_2 v_1} \\
&f_y = \frac{Bu_1 - Av_1}{u_1 v_2 - u_2 v_1},
\end{align*}
\]

Since we assumed \( \hat{u} \) and \( \hat{v} \) are nonparallel, \( \frac{u_1}{u_2} \neq \frac{v_1}{v_2} \implies u_1 v_2 - u_2 v_1 \neq 0 \). In conclusion, it is possible to recover \( \nabla f \big|_P \) from the given information and

\[
\nabla f \big|_P = \left\langle \frac{Av_2 - Bu_2}{u_1 v_2 - u_2 v_1}, \frac{Bu_1 - Av_1}{u_1 v_2 - u_2 v_1} \right\rangle.
\]

---

1One might interpret the quantity \( u_1 v_2 - u_2 v_1 \) as a two-dimensional analog of the cross product.