Problem 1.

\[ a_{n+1} = \frac{1}{2}(a_n + b_n); \quad b_{n+1} = \sqrt{a_n b_n} \]

(a) Show \( \{a_n\} \) and \( \{b_n\} \) converge quadratically to \( M \).

First, we claim that \( \{a_n\}, \{b_n\} \rightarrow M \).

Proof. Consider the case when \( a_0 > b_0 \). By induction,

\[ a_{n+1} = \frac{1}{2}(a_n + b_n) < \frac{1}{2}(a_n + a_n) = a_n \quad \forall \ n \in \mathbb{N} \]

Similarly for \( b_n \),

\[ b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n \quad \forall \ n \in \mathbb{N} \]

Also, for \( a_0 > b_0 \), \( (a_n + b_n)(a_n - b_n) > 0 \), so we can conclude

\[ 0 < (a_n + b_n)(a_n - b_n) \]
\[ 0 < a_n^2 - 2a_n b_n + b_n^2 \]
\[ 4a_n b_n < a_n^2 + 2a_n b_n + b_n^2 \]
\[ a_n b_n < \frac{1}{4}(a_n^2 + 2a_n b_n + b_n^2) \]
\[ \sqrt{a_n b_n} < \frac{1}{2}(a_n + b_n) \]
\[ b_{n+1} < a_{n+1} \quad \forall \ n \in \mathbb{N} \]

In summary, given that \( a_0 > b_0 \), \( a_n \) is strictly decreasing and \( b_n \) is strictly increasing for all \( n \).

Additionally, we have shown that \( a_n \) is bounded below by \( b_n \) and that \( b_n \) is bounded above by \( a_n \). Therefore \( \{a_n\} \) and \( \{b_n\} \) are both convergent and we write \( a_n \rightarrow A \) and \( b_n \rightarrow B \).

It remains to show that \( A = B \). This is clear since

\[ \lim_{n \to \infty} a_n = A \]
\[ = \lim_{n \to \infty} \frac{1}{2}(a_n + b_n) \]
\[ = \frac{1}{2}(A + B) \]
\[ \implies A = B = M \]
Next, we show that the convergence of the sequences is quadratic. If the difference of $a_n$ and $b_n$ converges quadratically to 0, then we can say that each sequence converges quadratically to $M$.

**Proof.** Briefly, assume that $a_n$ converges to $M$ slower than quadratically yet the difference of $a_n$ and $b_n$ still converges quadratically 0. Even if we assume that $b_n = M \forall n$, the difference will not converge quadratically to 0. Hence, we have a contradiction and see that quadratic convergence of $a_n - b_n \to 0$ implies quadratic convergence of each sequence individually. □

Taking the definition of the asymptotic error constant, we have order of convergence $\alpha$ if $\lambda$ exists in the limit and is finite. Taking $\alpha = 2$,

$$
\lim_{n \to \infty} \frac{|a_{n+1} - b_{n+1}|}{|a_n - b_n|^2} = \lim_{n \to \infty} \frac{|\frac{1}{2}(a_n + b_n) - \sqrt{a_nb_n}|}{|a_n - b_n|^2}
$$

$$
= \lim_{n \to \infty} \frac{\frac{1}{2}(\sqrt{a_n} - \sqrt{b_n})^2}{(\sqrt{a_n} + \sqrt{b_n})^2(\sqrt{a_n} - \sqrt{b_n})^2}
$$

$$
= \lim_{n \to \infty} \frac{1}{2(\sqrt{a_n} + \sqrt{b_n})^2}
$$

$$
= \frac{1}{2(2\sqrt{M})^2}
$$

$$
\lambda = \frac{1}{8M}
$$

Since the limit exists and is finite, we can conclude that the difference of $a_n$ and $b_n$ converges quadratically and therefore, each sequence alone converges quadratically. In the case that $b_0 > a_0$, we reach a similar result by the same methods.

(b) Taking $a_0 = 1 + x$ and $b_0 = 1 - x$ and expanding $a_n$ and $b_n$ about $x = 0$ for $n = 1, 2, 3, 4, 5$, we see that the first $2^n - 1$ terms in the series match exactly. That is,

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>coeffs</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>

(c) Yes, the expansions match exactly. Calculated in Mathematica,

$$
\frac{1}{M(1 + x, 1 - x)} \approx 1 + \frac{x^2}{4} + \frac{9x^4}{64} + \frac{25x^6}{256} + \frac{1225x^8}{16384} + \frac{3969x^{10}}{65536} + O[x]^{11}
$$

$$
\frac{1}{\pi} \int_0^{\pi} \frac{d\phi}{\sqrt{1 - x^2 \cos^2 \phi}} \approx 1 + \frac{x^2}{4} + \frac{9x^4}{64} + \frac{25x^6}{256} + \frac{1225x^8}{16384} + \frac{3969x^{10}}{65536} + O[x]^{11}
$$

**Problem 2.** Let $f(x) = e^{-\frac{1}{x^2}}$. The root is at zero and has infinite multiplicity.

(a) Where can we choose $x_0$ to guarantee convergence of Newton’s method?

The basin of attraction is $\{x_0 : |x_0| < 2\}$. 

2
Proof. Constructing Newton’s method, \( f'(x) = \frac{2e^{\frac{1}{x^2}}}{x^3} \) so

\[ x_{n+1} = x_n - \frac{x_n^3}{2} \]

In the basin of attraction, for some \( n \), we want every \( x_{n+1} \) to be closer to the root \( (p = 0) \) than \( x_n \). That is, we want \( |x_{n+1} - p| < |x_n - p| \) \( \implies |x_{n+1}| < |x_n| \).

\[
|\frac{x_n - x_n^3}{2}| < |x_n| \\
|\frac{x_n}{2}| |1 - x_n^2| < |x_n| \\
\implies |2 - x_n^2| < 2
\]

Decomposing the absolute value, we see \( 4 > x_n^2 > 0 \). Noting that \( \sqrt{x^2} \equiv |x| \),

\[ |x_n| < 2 \]

(b) If we start with \( x_0 = 1 \), estimate how many iterations it will take to come within \( 10^{-6} \) of the root.

We consider \( x_{n+1} = x_n - \frac{x_n^3}{2} \) and note that for a single iteration step \( \Delta n = 1 \) and \( \Delta x = -\frac{x_n^3}{2} \).

Writing as a differential equation,

\[ \frac{dx}{dn} = \frac{x^3}{2} \]

We can separate and solve, giving us

\[ \frac{1}{x^2} = n + c; \quad c = 1 \text{ since } x(n = 0) = 1 \]

This implies that for \( x = 10^{-6} \),

\[ n = 10^{12} - 1 \]

Problem 3. Assume having a simple processor capable only of addition, subtraction, multiplication, and halving (I believe this is accomplished by a simple bit-shift operation). Devise a Newton-based algorithm to find \( \sqrt{a} \).

Let’s construct a function \( f \) that has a root at \( \frac{1}{\sqrt{a}} \).

\[ f(x) = x^2 - a; \quad f\left(\frac{1}{\sqrt{a}}\right) = 0 \]

Setting up a Newton algorithm for fixed point root finding, \( f'(x) = -2x^{-3} \), so

\[ x_{n+1} = x_n - \frac{x_n^2 - a}{-2x_n^3} = x_n + \frac{x_n^3}{2} - \frac{ax_n^3}{2} \]

We iterate this so \( x_n \to \frac{1}{\sqrt{a}} \) and then multiply the result by \( a \) to get the desired value.
**Problem 4.** Generalize Newton's method for better convergence. i.e. take more terms in the Taylor Series.

First, Taylor expand the function $f(x_{n+1})$ about the previous iterate $x_n$ where $x_{n+1} = x_n + \Delta x$.

$$f(x_{n+1}) = f(x_n) + f'(x_n)\Delta x + \frac{f''(x_n)}{2} \Delta x^2 + \frac{f'''(x_n)}{6} \Delta x^3 + ...$$

Assume that once we add the correct $\Delta x$, $f(x_{n+1}) = 0$. Using the usual Newton definition for the second $\Delta x$

$$f(x_{n+1}) = 0 = f(x_n) + f'(x_n)\Delta x + \frac{f''(x_n)}{2} \left( \frac{f(x_n)}{f'(x_n)} \right)^2$$

Rearranging to solve for $\Delta x$, we obtain

$$\Delta x = -\frac{f(x_n) - f''(x_n) f(x_n)^2}{f'(x_n) - 2 f'(x_n)^3}$$

making the iterative equation

$$x_{n+1} = x_n - \frac{f(x_n) - f''(x_n) f(x_n)^2}{f'(x_n) - 2 f'(x_n)^3}$$

(1)

**Equation (1) exhibits cubic convergence.**

**Proof.** Following the same steps that we used to show quadratic convergence for the standard Newton's method, let $x_n = p + \varepsilon_n$ and $x_{n+1} = p + \varepsilon_{n+1}$.

$$p + \varepsilon_{n+1} = p + \varepsilon_n - \frac{f(x_n) - f''(x_n) f(x_n)^2}{f'(x_n) - 2 f'(x_n)^3}$$

Taylor expanding about $p$ using the following Mathematica command,

Assuming $\left[ f(p) = 0, \text{Simplify} \left[ \frac{-\text{Series}[f(x), \{x, p, 3\}] - \text{Series}[f(x), \{x, p, 3\}]^2}{2\text{Series}[f'(x), \{x, p, 3\}]^3} \right] \right]$

$$\varepsilon_{n+1} = x_n - p - (- (x_n - p)) + \frac{(x_n - p)^3 (3 f''''(p)^2 - f^{(3)}(p) f'(p))}{6 f'(p)^2} + O((x_n - p)^4)$$

which, noting the $x_n - p = \varepsilon_n$ simplifies to

$$\varepsilon_{n+1} = -\frac{\varepsilon_n^3 (3 f''''(p)^2 - f^{(3)}(p) f'(p))}{6 f'(p)^2} + O(\varepsilon_n^4)$$

We have shown that for any $n$,

$$\frac{\varepsilon_{n+1}}{\varepsilon_n^3} = \frac{(3 f''''(p)^2 - f^{(3)}(p) f'(p))}{6 f'(p)^2} + O(\varepsilon_n)$$

In the limit as $n \to \infty$, $O(\varepsilon_n) \to 0$ since $x_n$ is converging to $p$, so we have the definition of cubic convergence with asymptotic error constant $\frac{(3 f''''(p)^2 - f^{(3)}(p) f'(p))}{6 f'(p)^2}$. □