1. Complete the following problems from the textbook:
   • Section 3.2: 1e, 5b, 11, 21a, 23, 25, 28, 37ac, 39
   • Section 3.3: 3, 10ace, 20bd, 32ac
   • Section 3.4: 1ce, 4ac, 15

2. Prove that all positive numbers \( a, b \in \mathbb{R} \) satisfy

\[
(a + b) \left( \frac{1}{a} + \frac{1}{b} \right) \geq 4.
\]

(HINT: Use the Cauchy-Schwarz inequality, with two cleverly chosen vectors in \( \mathbb{R}^2 \).)

3. Suppose that \( \mathbf{x} = (x_1, x_2, \ldots x_n)^T \) is a vector whose components are the weights (in pounds) of \( n \) individuals randomly drawn from a population. In general, such a vector is called a random vector. Similarly, let \( \mathbf{y} = (y_1, y_2, \ldots y_n)^T \) be random vector containing the heights (in inches) of the same \( n \) individuals. Note that the pairs \( (x_i, y_i) \) consist of the weight and height of the \( i \)th individual.

Define the correlation between \( \mathbf{x} \) and \( \mathbf{y} \) to be

\[
\rho_{x,y} = \frac{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left( \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \right) \left( \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \right)}},
\]

where \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \) is the mean of \( \mathbf{x} \) and \( \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \) is the mean of \( \mathbf{y} \).

(a) Prove that

\[
\rho_{x,y} = \frac{\langle \mathbf{x} - \bar{x}, \mathbf{y} - \bar{y} \rangle}{\| \mathbf{x} - \bar{x} \| \| \mathbf{y} - \bar{y} \|},
\]

where \( \langle *, * \rangle \) is the dot product and \( \| * \| \) is the Euclidean norm.

(b) Suppose that a group of \( n = 5 \) individuals were sampled from a population, and each individual had their weight and height recorded. Let

\[
\mathbf{x} = \begin{pmatrix} 154 \\ 173 \\ 154 \\ 184 \\ 184 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 67 \\ 72 \\ 66 \\ 72 \\ 71 \end{pmatrix}.
\]

Find \( \rho_{x,y} \).

(c) Let \( \theta \) be the angle between \( \mathbf{x} = (x_1, x_2, \ldots x_n)^T \) and \( \mathbf{y} = (y_1, y_2, \ldots y_n)^T \). Prove that the correlation between two random vectors is zero—i.e., the random vectors are uncorrelated—if and only if \( \theta = 0 \).