1. Suppose that a random sample of size 12, taken from a $N(\mu, 3)$ distribution, results in a sample mean of $-5.7$. Give an 88% confidence interval for the true mean $\mu$.

2. Suppose that a random sample of size 12, taken from the $N(\mu, \sigma^2)$ distribution, results in a sample mean of $-5.7$ and a sample variance of 2.8. Give a 90% confidence interval for the true mean $\mu$.

3. Suppose that we have a random sample of size $n_1$ from the $N(\mu_1, \sigma^2)$ distribution and an independent random sample of size $n_2$ from the $N(\mu_2, \sigma^2)$ distribution. Suppose that $\mu_1$, $\mu_2$, and $\sigma^2$ are unknown. In this problem, we are going to develop a $100(1-\alpha)%$ confidence interval for the difference in means, $\mu_1 - \mu_2$.

Suppose that $\bar{X}_1$ and $\bar{X}_2$ are the two sample means and $S^2_1$ and $S^2_2$ are the two sample variances.

Both distributions have the same variance $\sigma^2$. $S^2_1$ and $S^2_2$ are independent unbiased estimators of this variance. To use all information for estimating the variance, you might consider averaging the two sample variances. However, if one sample size is larger than the other, the corresponding sample variance is probably a better estimator so we should give it more weight. We could for example consider a weighted average such as $(n_1 S^2_1 + n_2 S^2_2)/(n_1 + n_2)$. However, it will be mathematically convenient to instead use the following weighted average

$$S^2_p := \frac{(n_1 - 1)S^2_1 + (n_2 - 1)S^2_2}{n_1 + n_2 - 2}$$

which we refer to as the “pooled variance”.

(a) Find the distribution of $\frac{(n_1 + n_2 - 2)S^2_p}{\sigma^2}$.

(Hint: break it apart into an $S^2_1$ part and an $S^2_2$ part.)

(b) Find the distribution of $\frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{S^2_1}{n_1} + \frac{S^2_2}{n_2}}}$

(Hint: First consider the distribution of $\frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}}$ and make adjustments.)

(c) Use part (b) to derive a $100(1-\alpha)%$ confidence interval for $\mu_1 - \mu_2$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from the $unif(0, \theta)$ distribution where $\theta > 0$. Consider estimating $\theta$ with the maximum $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$.

(a) Find a simple function of $\theta$ and $X_{(n)}$ whose distribution is “$\theta$-free”.

(b) Use your part (a) to find a $100(1-\alpha)%$ confidence interval for $\theta$ based on $X_{(n)}$. 
5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $exp(rate = \lambda)$ distribution. Consider testing the hypotheses

$$H_0: \lambda = 3 \quad \text{versus} \quad H_1: \lambda > 3$$

using the rejection rule

“Reject $H_0$ in favor of $H_1$ if $X_{(1)} < c$.”

Here $X_{(1)}$ is the minimum value in the sample.

Find the constant $c$ so that the level of significance (the “size” of the test) is 0.05.