The cross product of two vectors in $\mathbb{R}^2$ is defined as the scalar

\[ \mathbf{v} \times \mathbf{w} = v_1w_2 - v_2w_1, \]

for $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$.

(a) Does the cross product define an inner product? Carefully explain which axioms are valid and which are not.

1. **Bilinearity:**
   \[
   (c\mathbf{v} + d\mathbf{w}) \times \mathbf{z} = (c\mathbf{v} + d\mathbf{w})_1w_2 - (c\mathbf{v} + d\mathbf{w})_2w_1 \\
   = cu_1v_2 + du_1w_2 - cu_2w_1 - du_2v_1 \\
   = c(u_1w_2 - u_2w_1) + d(u_1v_2 - u_2v_1) \\
   = c(\mathbf{v} \times \mathbf{w}) + d(\mathbf{w} \times \mathbf{z}).
   \]

   \[
   (\mathbf{v} + \mathbf{w}) \times \mathbf{z} = (\mathbf{v} + \mathbf{w})_1w_2 - (\mathbf{v} + \mathbf{w})_2w_1 \\
   = u_1(c\mathbf{v} + d\mathbf{w})_2 - u_2(c\mathbf{v} + d\mathbf{w})_2 \\
   = cu_1v_2 + du_1w_2 - cu_2w_1 - du_2v_1 \\
   = c(u_1w_2 - u_2w_1) + d(u_1v_2 - u_2v_1) \\
   = c(\mathbf{v} \times \mathbf{w}) + d(\mathbf{w} \times \mathbf{z}).
   \]

   \[
   \Rightarrow \text{ Bilinearity holds}
   \]

2. **Symmetry:**
   Consider $\mathbf{v} = (0, 1)$ and $\mathbf{w} = (1, 0)$.
   Then
   \[
   \mathbf{v} \times \mathbf{w} = 1 \cdot 1 - 0 \cdot 0 = 1 \neq -1 = 0 \cdot 0 - 1 \cdot 1 = \mathbf{w} \times \mathbf{v}.
   \]

   \[
   \Rightarrow \text{ Symmetry fails}
   \]

3. **Positivity:**
   Consider $\mathbf{v} = (0, 1)$ for any nonzero vector.
   Then
   \[
   \mathbf{v} \times \mathbf{v} = 0 \cdot 1 - 1 \cdot 0 = 0,
   \]
   \[
   \Rightarrow \text{ By (2) and (3), cross product not an inner product}
   \]

(b) Prove that $\mathbf{v} \times \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\sin \theta$, where $\theta$ denotes the angle for $\mathbf{v}$ to $\mathbf{w}$.

Clearly true if $\mathbf{v} = \mathbf{w}$ or $\mathbf{v} = -\mathbf{w}$, so assume not.
Then
\[
\sin^2 \theta = 1 - \cos^2 \theta = 1 - \langle \mathbf{v}, \mathbf{w} \rangle^2
\]
\[
= \frac{\|\mathbf{v}\|^2\|\mathbf{w}\|^2 - \langle \mathbf{v}, \mathbf{w} \rangle^2}{\|\mathbf{v}\|^2\|\mathbf{w}\|^2}
\]
Note that
\[
\langle \mathbf{v}, \mathbf{w} \rangle^2 = (\mathbf{v}_1, \mathbf{w}_2)(\mathbf{w}_1, \mathbf{v}_2) = v_1w_2 - v_2w_1 = w_1v_2 - w_2v_1
\]
\[
= v_1w_2 + v_1w_2 + v_2w_2 + v_2w_2 - 2w_1v_2w_1
\]
\[
= (v_1^2 + v_2^2)(w_1^2 + w_2^2) - (v_1w_1 + v_2w_2)^2
\]
So
\[
\sin^2 \theta = \frac{\|\mathbf{v}\|^2\|\mathbf{w}\|^2 - \langle \mathbf{v}, \mathbf{w} \rangle^2}{\|\mathbf{v}\|^2\|\mathbf{w}\|^2}
\]
\[
\Rightarrow \sin \theta = \frac{\|\mathbf{v}\|^2\|\mathbf{w}\|^2 - \langle \mathbf{v}, \mathbf{w} \rangle^2}{\|\mathbf{v}\|^2\|\mathbf{w}\|^2}
\]

(c) Prove that $\mathbf{v} \times \mathbf{w} = 0 \iff \mathbf{v}, \mathbf{w}$ are parallel.

By (b), $\mathbf{v} \times \mathbf{w} = 0 \iff \sin \theta = 0$ or $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0} \iff \mathbf{v}, \mathbf{w}$ are parallel.

(d) Show that $\|\mathbf{v} \times \mathbf{w}\|$ equals the area of the parallelogram defined by $\mathbf{v}$ and $\mathbf{w}$.

Length is $\|\mathbf{v}\|\|\mathbf{w}\|$, height is given by $\sin \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$.
3.2.25 Let \( V \) be an inner product space and \( v \in V \) a fixed element. Prove that the set of all vectors \( w \in V \) that are orthogonal to \( v \) forms a subspace of \( V \).

Consider 2 vectors \( \bar{u}, \bar{w} \in V \) that are orthogonal to \( v \), and \( c \in \mathbb{R} \).

1. Thus by definition, \( \langle \bar{u}, v \rangle = 0 = \langle \bar{w}, v \rangle \).
2. So \( \langle \bar{u} + \bar{w}, v \rangle = \langle \bar{u}, v \rangle + \langle \bar{w}, v \rangle \) (by Bilinearity) = \(0 + 0 = 0\).

So \( \bar{u} + \bar{w} \) is orthogonal to \( v \), so the set is closed under addition.

3. And \( \langle c\bar{u}, v \rangle = c \langle \bar{u}, v \rangle \) (by Bilinearity) = \(c \cdot 0 = 0\).

So \( c\bar{u} \) is orthogonal to \( v \), so the set is closed under scalar multiplication.

By 1 and 2, the set is a subspace of \( V \).

3.2.39 Prove that \( |\bar{v} - \bar{v}^\prime| > |\bar{v} - \bar{v}^\prime| - |\bar{v} - \bar{v}^\prime| \).

Interpret this result pictorially

Write \( \bar{v} = (\bar{v} - \bar{v}^\prime) + \bar{v}^\prime \)

So \( |\bar{v}| = |(\bar{v} - \bar{v}^\prime) + \bar{v}^\prime| \leq |\bar{v} - \bar{v}^\prime| + |\bar{v}^\prime| \),

\( \Rightarrow |\bar{v} - \bar{v}^\prime| > |\bar{v} - \bar{v}^\prime| - |\bar{v}^\prime| \) \hspace{1cm} \text{(by Triangle Inequality)}

Similarly, write \( \bar{v}^\prime = (\bar{v} - \bar{v}^\prime) + \bar{v} \)

Then \( |\bar{v}^\prime| = |(\bar{v} - \bar{v}^\prime) + \bar{v}| \leq |\bar{v} - \bar{v}^\prime| + |\bar{v}| \) by Triangle Inequality,

\( \Rightarrow |\bar{v} - \bar{v}^\prime| = |\bar{v} - \bar{v}^\prime| > |\bar{v}^\prime| - |\bar{v} - \bar{v}^\prime| \) \hspace{1cm} \text{(by Triangle Inequality)}

Combining 1 and 2, we have,

\( |\bar{v} - \bar{v}^\prime| > |\bar{v} - \bar{v}^\prime| - |\bar{v} - \bar{v}^\prime| \).

Pictorially, the 3rd side of a triangle is at least as long as the difference between the other 2 sides.
Which two of the vectors $\mathbf{u} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ are closest to each other in distance for

(a) the Euclidean norm?

$\|\mathbf{u} - \mathbf{v}\|_2 = \|\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}\|_2 = \sqrt{9 + 4 + 0} = \sqrt{13}$

$\|\mathbf{u} - \mathbf{w}\|_2 = \|\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}\|_2 = \sqrt{4 + 4 + 4} = \sqrt{12}$

$\|\mathbf{v} - \mathbf{w}\|_2 = \|\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}\|_2 = \sqrt{1 + 16 + 4} = \sqrt{21}$

$\Rightarrow \mathbf{u}, \mathbf{w}$ closest in 2-norm

(b) the $\infty$ norm?

$\|\mathbf{u} - \mathbf{v}\|_\infty = \|\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}\|_\infty = \max \{|-3|, |-2|, |0|\} = 3$

$\|\mathbf{u} - \mathbf{w}\|_\infty = \|\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}\|_\infty = \max \{|-2|, |-2|, |2|\} = 2$

$\|\mathbf{v} - \mathbf{w}\|_\infty = \|\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}\|_\infty = \max \{|1|, |4|, |2|\} = 4$

$\Rightarrow \mathbf{u}, \mathbf{w}$ closest in $\infty$-norm

(c) the 1-norm?

$\|\mathbf{u} - \mathbf{v}\|_1 = \|\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}\|_1 = |3| + |2| + |0| = 5$

$\|\mathbf{u} - \mathbf{w}\|_1 = \|\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}\|_1 = |2| + |2| + |2| = 6$

$\|\mathbf{v} - \mathbf{w}\|_1 = \|\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}\|_1 = |1| + |4| + |2| = 7$

$\Rightarrow \mathbf{u}, \mathbf{v}$ closest in 1-norm
#2 Prove that all positive numbers \(a, b \in \mathbb{R}\) satisfy
\[
(a+b) \left( \frac{1}{a} + \frac{1}{b} \right) \geq 4.
\]

Consider positive numbers \(a, b \in \mathbb{R}\).
Let \( \mathbf{v} = \left( \frac{1}{a} \right) \) and \( \mathbf{w} = \left( \frac{1}{b} \right) \). [Which exist, since \(a, b > 0\).]

Since \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^2\), they satisfy the Cauchy-Schwarz inequality.
So
\[
(a+b) \left( \frac{1}{a} + \frac{1}{b} \right) = \| \mathbf{v} \|^2 \| \mathbf{w} \|^2 
\]
\[
\geq (\mathbf{v} \cdot \mathbf{w})^2 \quad \text{(by C-S inequality)}
\]
\[
= \left( \sqrt{a} \cdot \sqrt{\frac{1}{a}} + \sqrt{b} \cdot \sqrt{\frac{1}{b}} \right)^2
\]
\[
= (1+1)^2 = 4. \quad \text{//}
\]
\[
\text{In fact for positive numbers } a_1, \ldots, a_n \in \mathbb{R},
\]
\[
(a_1 + \ldots + a_n) \left( \frac{1}{a_1} + \ldots + \frac{1}{a_n} \right) \geq n^2
\]

#3 Suppose that \( \mathbf{x} = (x_1) \) is a vector whose components are the weights (in pounds) of \(n\) individuals randomly drawn from a population. In general, such a vector is called a "random vector." Similarly, let \( \mathbf{y} = (y_1) \) be a random vector containing the heights (in inches) of the same \(n\) individuals. Note that the pairs \((x_i, y_i)\) consist of the weight and height of the \(i\)th individual. Define the "correlation" between \(\mathbf{x}\) and \(\mathbf{y}\) to be
\[
\rho_{xy} = \frac{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2}}
\]
where \(\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i\) is the mean of \(\mathbf{x}\), and \(\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i\) is the mean of \(\mathbf{y}\).

(a) Prove that
\[
\rho_{xy} = \frac{\langle \mathbf{x} - \bar{x}, \mathbf{y} - \bar{y} \rangle}{\| \mathbf{x} - \bar{x} \| \| \mathbf{y} - \bar{y} \|}
\]
where \(\langle \mathbf{x}, \mathbf{y} \rangle\) is the dot product, and \(\| \mathbf{x} \|\) is the Euclidean norm.

\[
\rho_{xy} = \frac{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2}}
\]
\[
= \frac{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2}}
\]
\[
= \frac{\langle \mathbf{x} - \bar{x}, \mathbf{y} - \bar{y} \rangle}{\| \mathbf{x} - \bar{x} \| \| \mathbf{y} - \bar{y} \|} \quad \text{//}
\]
(b) Suppose that a group of \( n = 5 \) individuals were sampled from a population, and each individual had their weight and height recorded. Let

\[
\mathbf{x} = \begin{pmatrix} 154 \\ 173 \\ 154 \\ 154 \\ 184 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 67 \\ 72 \\ 64 \\ 72 \\ 71 \end{pmatrix}.
\]

Find \( s_{xy} \).

To compute, need \( \bar{x} \) and \( \bar{y} \). Here \( \bar{x} = \frac{1}{5} (154 + 173 + 154 + 154 + 184) = 169.8 \)

\( \bar{y} = \frac{1}{5} (67 + 72 + 64 + 72 + 71) = 69.6 \)

So \( s_{xy} = \frac{\sum (x_i - \bar{x}) (y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2} \sqrt{\sum (y_i - \bar{y})^2}} \)

\[
= \frac{\sum (x_i - \bar{x}) (y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2} \sqrt{\sum (y_i - \bar{y})^2}} = \frac{41.08 + 7.68 + 56.88 + 34.08 + 19.88}{\sqrt{912.8} \sqrt{33.2}}
\]

\[
= 0.9168
\]

(c) Let \( \Theta \) be the angle between \( \mathbf{x}^* = \begin{pmatrix} x_1 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix} \) and \( \mathbf{y}^* = \begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix} \).

Prove that the correlation between these random vectors is zero — i.e., the random vectors are “uncorrelated” if and only if \( \Theta = \frac{\pi}{2} \).

\( s_{x^*y^*} = 0 \iff \frac{\mathbf{x}^* \cdot \mathbf{y}^*}{\| \mathbf{x}^* \| \| \mathbf{y}^* \|} = 0 \)

\[
\frac{\mathbf{x}^* \cdot \mathbf{y}^*}{\| \mathbf{x}^* \| \| \mathbf{y}^* \|} = 0 \quad (\text{since} \ \| \mathbf{x}^* \| = 0 = \| \mathbf{y}^* \|) = 0.
\]

\( \cos \Theta = 0, \) where \( \Theta \) is the angle between \( \mathbf{x}^* \) and \( \mathbf{y}^* \).

\( \Theta = \frac{\pi}{2} \).