APPMT 5440: Applied Analysis I
Problems for Problem Set Six (Due Friday, October 23rd)

1. (H & N 2.13) Consider the scalar initial value problem,
\[ \dot{u}(t) = |u(t)|^\alpha \]
\[ u(0) = 0. \]
Show that the solution is unique if \( \alpha \geq 1 \), but not if \( 0 \leq \alpha < 1 \).
(Note that \( 0^0 \) is undefined so there is no solution when \( \alpha = 0 \).)

2. (H & N 2.14) Suppose that \( f(t, u) \) is a continuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that
\[ |f(t, u) - f(t, v)| \leq K|u - v| \quad \forall \ t, u, v \in \mathbb{R}. \]
Also suppose that
\[ M = \sup\{|f(t, u_0)| : |t - t_0| \leq T\}. \]
(a) Prove that the solution \( u(t) \) of the initial value problem
\[ \dot{u}(t) = f(t, u), \quad u(t_0) = u_0 \]
satisfies the estimate
\[ |u(t) - u_0| \leq MTe^{KT} \quad \text{for} \quad |t - t_0| \leq T. \]
(b) Explicitly check this estimate for the initial value problem
\[ \dot{u} = Ku, \quad u(t_0) = u_0. \]

3. Let \( k : [a, b] \times [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) be continuous, and suppose that
\[ \sup_{a \leq x \leq b} \left\{ \int_a^b |k(x, y)| \, dy < 1 \right\}. \]
Let \( K \) be the operator
\[ Kf(x) := \int_a^b k(x, y) f(y) \, dy. \]
Show that \( T^n f_0 := g + Kg + K^2g + \cdots + K^{n-1}g + K^n f_0 \)
converges uniformly to \( \sum_{n=0}^\infty K^n g \) on \([a, b]\) for any \( f_0 \in C([a, b])\).

4. Solve for \( f(x) \).
(a) \( f(x) = \frac{5}{6}x + \frac{1}{2} \int_0^1 xyf(y) \, dy \)
(b) \( f(x) = e^x + \lambda \int_0^1 2e^{x+y}f(y) \, dy \).

5. The following integral equation for \( f : [-a, a] \to \mathbb{R} \) arises in a model of the motion of gas particles on a line:
\[ f(x) = 1 + \frac{1}{\pi} \int_{-a}^a \frac{1}{1 + (x - y)^2} f(y) \, dy \quad \text{for} \quad -a \leq x \leq a. \]
Prove that this equation has a unique bounded, continuous solution for every \( 0 < a < \infty \).
Prove that the solution is nonnegative. What can you say if \( a = \infty \)?