30. \( V = \int_{0}^{1} \int_{1}^{1} (3y^2 - x^2 + 2) \, dA = \int_{-1}^{1} \left( \int_{3y^2 - x^2 + 2}^{1} dy \right) \, dx = \int_{-1}^{1} \left[ y^3 - x^2y + 2y \right]_{y=1}^{y=2} \, dx \)
\( = \int_{-1}^{1} \left[ (12 - 2x^2) - (3 - x^2) \right] \, dx = \int_{-1}^{1} (9 - x^2) \, dx = [9x - \frac{1}{3}x^3]_{-1}^{1} = \frac{26}{3} + \frac{26}{3} = \frac{52}{3} \)

34. The cylinder intersects the \( xy \)-plane along the line \( x = 4 \), so in the first octant, the solid lies below the surface \( z = 16 - x^2 \) and above the rectangle \( R = [0, 4] \times [0, 5] \) in the \( xy \)-plane.
\( V = \int_{0}^{5} \left( \int_{0}^{4} (16 - x^2) \, dx \right) \, dy = \int_{0}^{5} \left( 16y - \frac{4}{3}x^3 \right) \, dy = [16y - \frac{4}{3}x^3]_{0}^{5} = (64 - \frac{64}{3}) - 0 = \frac{640}{3} \)

Section 12.2

40. Because the region of integration is
\( D = \{ (x, y) \mid 0 \leq x \leq \sqrt{4 - y^2}, -2 \leq y \leq 2 \} \)
\( = \{ (x, y) \mid -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, 0 \leq x \leq 2 \} \)
we have
\( \int_{-2}^{2} \int_{0}^{\sqrt{4 - x^2}} f(x, y) \, dy \, dx = \int_{0}^{2} \int_{-\sqrt{4 - x^2}}^{\sqrt{4 - x^2}} f(x, y) \, dy \, dx. \)

42. Because the region of integration is
\( D = \{ (x, y) \mid \arctan x \leq y \leq \frac{\pi}{4}, 0 \leq x \leq 1 \} \)
\( = \{ (x, y) \mid 0 \leq x \leq \tan y, 0 \leq y \leq \frac{\pi}{4} \} \)
we have
\( \int_{0}^{\arctan x} \int_{0}^{\pi/4} f(x, y) \, dy \, dx = \int_{0}^{\tan y} \int_{0}^{\pi/4} f(x, y) \, dx \, dy. \)

46. \( \int_{0}^{1} \int_{0}^{1} e^{x/y} \, dy \, dx = \int_{0}^{1} \int_{0}^{y} e^{x/y} \, dx \, dy = \int_{0}^{1} \left[ ye^{x/y} \right]_{x=0}^{x=y} \, dy \)
\( = \int_{0}^{1} (e - 1) y \, dy = \frac{1}{2} (e - 1) \left( y^2 \right)_{0}^{1} \)
\( = \frac{1}{2} (e - 1) \)
2a. We want the first-order Taylor approximation (we'll call it $T_1(x,y)$) to the function $f(x,y) = x^4y^3$, about the point $(1,2)$. We need the following:

- $f(1,2) = 8$.
- $f_x(x,y) = 4x^3y^3$; $f_x(1,2) = 32$.
- $f_y(x,y) = 3x^4y^2$; $f_y(1,2) = 12$.

Therefore $T_1(x,y) = 8 + 32(x-1) + 12(y-2)$.

$T_1(1.1, 2.2) = 8 + 3.2 + 2.4 = 13.6$.

Note: actual $f(1.1, 2.2) = 15.5897$.

Error $\approx$ 2 units.

6) To bound the error in $T_1(x,y)$, we'll need "worst-case" info from second-order derivatives of $f$:

- $f_{xx} = 12x^2y^3$; worst case $= 12(1.1)^2(2.2)^3 \approx 155$.
- $f_{xy} = 12x^3y^2$; worst case $= 12(1.1)^3(2.2)^2 \approx 78$.
- $f_{yy} = 6x^4y^1$; worst case $= 6(1.1)^4(2.2) \approx 20$.

Since our second derivatives are bounded above by $M=155$ on the interval $(x$ and $y)$ in question, our error is bounded as follows:

$$E_1(x,y) \leq \frac{155}{2} \left( |x-1|^2 + 2|x-1| |y-2| + |y-2|^2 \right)$$

$$\leq \frac{155}{2} \left( 0.01 + 2(0.1)(0.2) + 0.04 \right) = 6.98.$$  

This is consistent as an upper bound with our error here.
\[ T_2(x, y) = T_1(x, y) + \frac{1}{2!} (f_x(1/2, 1/2) (x-1)^2 + 2(f_{xy}(1/2, 1/2) (x-1)(y-2)) + f_y(1/2, 1/2) (y-2)^2) \]

\[ = 13.6 + \frac{1}{2} (0.96 + 1.92 + 0.48) \]

\[ = 15.28 \quad \text{(recall actual } f(1.1, 2.2) = 15.59; \text{ error } \approx 0.3) \]

2d) To bound our second-order approximation's error, we need worst-case info from our third derivatives:

\[ f_{xxx} = 24xy^3; \quad \text{worst case } = 24(1.1)(2.2)^3 \approx 281.1 \]

\[ f_{xxy} = 36x^2y^2; \quad \text{worst case } = 36(1.1)^2(2.2)^2 \approx 211 \]

\[ f_{xyy} = 24x^3y; \quad \text{worst case } = 24(1.1)^3(2.2) \approx 70.3 \]

\[ f_{yyy} = 6x^4; \quad \text{worst case } = 6(1.1)^4 \approx 8.8 \]

\[ \Rightarrow M = 282 \]

\[ E_2(x, y) \leq \frac{M}{3!} (|x-1|^3 + 3|x-1|^2|y-2| + 3|x-1|(y-2)^2 + |y-2|^3) \]

\[ \leq \frac{282}{6} (0.001 + 0.006 + 0.012 + 0.008) = 1.27 \]

Again, this a consistent upper bound with the actual error found in 2c.
When we want to find \( x, y \) that satisfy

\[
\begin{align*}
  f(x, y) &= 0 \\
  g(x, y) &= 0
\end{align*}
\]

(about an initial "guess" \((x_0, y_0)\))

Simultaneously, we linearize \( f \) into \( L_f(x, y) \)
and \( g \) into \( L_g(x, y) \). The following two expressions:

\[
\begin{align*}
  L_f(x, y) &= 0 \\
  L_g(x, y) &= 0
\end{align*}
\]

should describe lines in the \( x-y \) plane. If these

lines are not parallel:

\[ L_g(x, y) = 0 \]

then the lines intersect at a new guess to
the root of the nonlinear system \((x_0, y_0)\).

From here, we linearize again, creating \( L_f(x, y) \)
and \( L_g(x, y) \) from a 1st-order Taylor approximation
about this new location, and we repeat the process
as desired.
After linearization, we're solving the following problem:
\[ f + f_x(x_1 - x_0) + f_y(y_1 - y_0) = 0 \]
\[ g + g_x(x_1 - x_0) + g_y(y_1 - y_0) = 0 \]

for \( x_1, y_1 \); all other quantities are known.

From the 1st equation, we can rearrange to obtain
\[ (x_1 - x_0) = -\frac{f - f_y(y_1 - y_0)}{f_x}. \]

Substituting this into the 2nd equation,
\[ g + \frac{-g_x f - g_x f_y(y_1 - y_0) + g_y(y_1 - y_0)}{f_x} = 0 \]
\[ f_x g - g_x f - g_x f_y(y_1 - y_0) + g_y f_x(y_1 - y_0) = 0 \]
\[ f_x g - g_x f = g_x f_y(y_1 - y_0) - g_y f_x(y_1 - y_0) \]
\[ y_1 = \frac{f_x g - g_x f}{g_x f_y - g_y f_x} + y_0, \text{ as desired.} \]
For the x-component, we substitute in our identity of $(g_1 - y_0)$:

\[ g + g_x (x - x_0) + g_y \left( \frac{f_x g_y - g_x f_y}{g_y f_x - g_x f_y} \right) = 0 \]

\[ (x_1 - x_0) = -g_y g_x - g_y \left( \frac{f_x g_y - g_x f_y}{g_y f_x - g_x f_y} \right) \]

\[ = -g_y g_x - \left( \frac{g_x f_y - g_y g_x f_y}{(g_x)^2 f_x - g_x g_y f_x} \right) \]

\[ = -\frac{1}{f_x} \left( g_y f_y - g_y f_y \right) \]

\[ = -\left( \frac{f_x g_y - g_x f_y}{g_y f_x - g_x f_y} \right) \]

\[ \Rightarrow \quad x_1 = x_0 + \frac{g_y f_y - g_x f_x}{f_x g_y - f_y g_x} \]
3c.
\[ f_x = -2xy \quad f_y = 1 - x^2 \quad f = y (1 - x^2) \]
\[ g_x = -y \quad g_y = -x \quad g = 2 - xy \]

\[ x_1 = x_0 + \frac{g_y f_x - f_y g_x}{f_x g_y - f_y g_x} = x_0 + \frac{(2-x_0 y_0)(1-x_0^2) - y_0(1-x_0^2)(-x_0)}{(-2x_0 y_0)(-x_0) - (1-x_0^2)(-y_0)} \]

\[ y_1 = y_0 + \frac{g_y f_x - f_y g_x}{f_x g_y - f_y g_x} = y_0 + \frac{y_0 (1-x_0^2)(-y_0) - (2-x_0 y_0)(-2x_0 y_0)}{(-2x_0 y_0)(-x_0) - (1-x_0^2)(-y_0)} \]

With \((x_0, y_0) = (2, 1)\), we have

\[ x_1 = 0.8, \quad y_1 = 1.6. \]

If we use \((x_1, y_1)\) as guesses for another iteration (replace \(x_0, y_0\) with \(x_1, y_1\) in equations above) we obtain

\[ x_2 = 1.0744, \quad y_2 = 1.9512. \]

Although not required, if you continue, you get

\[ x_3 = 1.0010, \quad y_3 = 1.9949 \]

\[ x_4 = 0.999998, \quad y_4 = 1.9999991. \]

We are indeed converging quadratically to the actual root at \((1, 2)\).
4a. \[ \int_{0}^{1} \int_{0}^{1} \frac{xy(x^2-y^2)}{(x^2+y^2)^3} \, dy \, dx \]

Let \( u = x^2 + y^2 \), \( \frac{du}{dy} = 2y \), \( y \, dy = \frac{1}{2} \, du \).

When \( y = 0 \), \( u = x^2 \), when \( y = 1 \), \( u = x^2 + 1 \).

\[ = \frac{1}{2} \int_{0}^{1} \int_{x^2}^{x^2+1} \frac{x(-u+2x^2)}{u^3} \, du \, dx \]

\[ = \frac{1}{2} \int_{0}^{1} \int_{x^2}^{x^2+1} \left( \frac{-x}{u^2} + \frac{2x^3}{u^3} \right) \, du \, dx \]

\[ = \frac{1}{2} \int_{0}^{1} \left[ \frac{x^{2+1}}{u} - \frac{x^3}{u^2} \right]_{x^2}^{x^2+1} \, dx \]

\[ = \frac{1}{2} \int_{0}^{1} \left( \frac{x}{x^2+1} - \frac{x^3}{(x^2+1)^2} - \frac{1}{x} \right) \, dx = \frac{1}{5} \]

(using Mathematica)

Note: be extremely on guard if two infinite integrands cancel like this!
\[
\text{46. } \iint_{D} \frac{x y (x^2 - y^2)}{(x^2 + y^2)^3} \, dx \, dy
\]

Again, let \( u = x^2 + y^2 \). \( \frac{du}{dx} = 2x \), \( x \, dx = \frac{1}{2} \, du \).

When \( x = 0 \), \( u = y^2 \). When \( x = 2 \), \( u = 4 + y^2 \).

\[
= \frac{1}{2} \int_{0}^{1} \int_{y^2}^{4+y^2} \frac{y (u-2y^2)}{u^3} \, du \, dy
\]

\[
= \frac{1}{2} \int_{0}^{1} \int_{y^2}^{4+y^2} \frac{y}{u} - \frac{2y^3}{u^3} \, du \, dy
\]

\[
= \frac{1}{2} \int_{0}^{1} \left[ \frac{1}{u^2} \left( \frac{y}{u} + \frac{y^3}{u^2} \right) \right] \, dy
\]

\[
= \frac{1}{2} \int_{0}^{1} \left( \frac{1}{4+y^2} + \frac{y^3}{(4+y^2)^2} + \frac{1}{y} \right) \, dy \to 0 \quad \text{(using Mathematica)}
\]
4c. Although the dy dx integral returned a value of 0.2 and the dx dy integral a value of \(-\frac{1}{20}\), this does not contradict Fubini, since Fubini's theorem requires absolute integrability - which this function lacks on the interval in question.

Observe:

\[ R_1 \cup R_2 = R \], our entire region.

Let's just try to integrate our \(|f(x,y)|\) function in region \(R_1\).

\[ -\int_0^1 \int_0^y \frac{xy(x^2-y^2)}{(x^2+y^2)^3} \, dx \, dy = -\frac{1}{2} \int_0^1 \int_y^{y^2} \frac{y^2}{u^2} \, du \, dy \]

\[ = -\frac{1}{2} \int_0^1 \left[ \frac{2y^2}{u} - \frac{y^3}{u^3} \right] \, du \, dy = -\frac{1}{2} \int_0^1 \left[ \frac{2y^2}{u} - \frac{y^3}{u^3} \right] \, dy \]

\[ = -\frac{1}{2} \int_0^1 \frac{1}{y^2} \frac{y-2}{4} \, dy = \frac{1}{18} \int_0^1 \frac{1}{y} \, dy. \]

This is a divergent improper integral. Thus, the function is not absolutely integrable over region \(R_1\), and thus it is not abs. integrable over \(R_1\).