Problem 1. Prove the recursion relations for the first derivatives of Chebyshev and Legendre polynomials.

(a)

\[ \frac{T'_{n+1}}{n+1} - \frac{T'_{n-1}}{n-1} = 2T_n \]

Proof. Recall the expression \( T_n(x) = \cos(n \arccos(x)) \) for \( n \geq 0 \) for the standard weight function \( w(x) = \frac{1}{\sqrt{1-x^2}} \). Then \( T'_n(x) = \frac{n}{\sqrt{1-x^2}} \sin(n \arccos(x)) \)

\[
\frac{T'_{n+1}}{n+1} - \frac{T'_{n-1}}{n-1} = \frac{1}{n+1} \frac{n+1}{\sqrt{1-x^2}} \sin((n+1) \arccos(x)) - \frac{1}{n-1} \frac{n-1}{\sqrt{1-x^2}} \sin((n-1) \arccos(x))
\]

\[
= \frac{1}{\sqrt{1-x^2}} [\sin((n+1) \arccos(x)) - \sin((n-1) \arccos(x))]\]

Trig identity: \( \sin(a) - \sin(b) = 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \)

\[
= \frac{2}{\sqrt{1-x^2}} \left[ \cos\left(\frac{(n+1) \arccos(x) + (n-1) \arccos(x)}{2}\right) \sin\left(\frac{(n+1) \arccos(x) - (n-1) \arccos(x)}{2}\right) \right]
\]

\[
= \frac{2}{\sqrt{1-x^2}} \cos(n \arccos(x)) \sin(n \arccos(x))
\]

It's easy to show with a triangle that \( \sin(n \arccos(x)) = \sqrt{1-x^2} \)

\[ = 2T_n(x) \]

\[ \square \]

(b)

\[ P'_{n+1} - P'_{n-1} = (2n+1)P_n \]
Proof. Recall that

\[ P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1 - x^2)^n] \]

Then

\[ P'_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [-2nx(1 - x^2)^{n-1}] \]

Plugging in the above expression with \( n+1 \) and \( n-1 \),

\[
P'_{n+1}(x) - P'_{n-1}(x) = \frac{(-1)^{n+1}}{2^{n+1} (n+1)!} \frac{d^{n+1}}{dx^{n+1}} [-2(n+1)x(1 - x^2)^n] - \frac{(-1)^{n-1}}{2^{n-1} (n-1)!} \frac{d^n}{dx^n} [(1 - x^2)^{n-1}] \\
= \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1 - x^2)^n - 2nx^2(1 - x^2)^{n-1}] - \frac{(-1)^{n-1}}{2^{n-1} (n-1)!} \frac{d^n}{dx^n} [(1 - x^2)^{n-1}] \\
= P_n(x) + \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [-2nx^2(1 - x^2)^{n-1}] + \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [2n(1 - x^2)^{n-1}] \\
= P_n(x) + \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [-2nx^2(1 - x^2)^{n-1} + 2n(1 - x^2)^{n-1}] \\
= P_n(x) + \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [2n(1 - x^2)^{n-1}(-x^2 + 1)] \\
= (1 + 2n)P_n(x)
\]
2. Numerical convergence rates

First, I consider a plot of the max norm error on $[-1, 1]$ for a few cut-off points.

We find that in max norm, Chebyshev has a much higher rate of convergence than Taylor. Here I plot the error between analytic arctan and two approximations, Chebyshev and Taylor, using the first twelve nonzero coefficients.
This is a very slow max norm error convergence for Taylor, which is to be expected given that we are evaluating right at the boundary of convergence. We can analyze the rate more precisely with a plot of error in current term over error in previous term.

The slopes of these plots tell us that Chebyshev is linearly convergent, and Taylor is sublinearly convergent.

3. **Numerical Chebyshev Coefficients**

First we consider \( N = 16 \). The nonzero coefficients are:

<table>
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<tr>
<th>True coefficient</th>
<th>Approximate coefficient</th>
<th>Magnitude of difference ( \times 10^{-9} )</th>
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<td>0.828427124746190</td>
<td>0.828427124746116</td>
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<td>0.000545001543895</td>
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<tr>
<td>-0.00001197079759</td>
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<td>0.872010000000178</td>
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</tbody>
</table>

Then, with \( N = 32 \) we find the nonzero coefficients to be
<table>
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</table>

4. HERMITE POLYNOMIALS

Hermite polynomials can be defined by the Rodrigues' Formula

\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \]

a. Explicit definitions. Given this formulation,

\[ H_0(x) = (-1)^0 e^{x^2} e^{-x^2} = 1 \]

\[ H_1(x) = (-1)^1 e^{x^2} \left( -2xe^{-x^2} \right) = 2x \]

\[ H_2(x) = (-1)^2 e^{x^2} \frac{d}{dx} \left( (4x^2 - 2)e^{-x^2} \right) = 4x^2 - 2 \]

b. Orthogonality.

\[ (H_m, H_n) = \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} \, dx \]

\[ = \int_{-\infty}^{\infty} H_m(x) e^{x^2} \left[ (-1)^n e^{x^2} \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right] \, dx \]

\[ (-1)^n (H_m, H_n) = \int_{-\infty}^{\infty} H_m(x) \left[ \frac{d^n}{dx^n} e^{-x^2} \right] \, dx \]

Do integration by parts, with \( u = H_m(x) \), \( dv = \frac{d^n}{dx^n} e^{-x^2} \, dx \). So \( v = \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \) and \( du = \frac{d}{dx} H_m(x) \, dx \), which is a polynomial with leading coefficient \( 2^m n \). The \( 2^m \) from the leading term of \( H_m \), and the \( n \) from the power rule. So the second derivative of \( H_m \) would have leading term \( 2^m n(n-1) \).

\[ \int_{-\infty}^{\infty} H_m(x) \left[ \frac{d^n}{dx^n} e^{-x^2} \right] \, dx = H_m(x) \left[ \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} H_m(x) \left[ \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right] \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{d}{dx} H_m(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \, dx \]
• if \( m = n \), we iterate integration by parts \( n \) times, until

\[
\int_{-\infty}^{\infty} H_m(x) \left[ \frac{d^n}{dx^n} e^{-x^2} \right] dx = (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{dx^n} H_m(x) e^{-x^2} dx
\]

\[
= (-1)^n \int_{-\infty}^{\infty} 2^n n! e^{-x^2} dx
\]

\[
= (-1)^n 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx
\]

\[
= (-1)^n \sqrt{\pi} 2^n n!
\]

and the \((-1)^n\) cancels from both sides of the equation.

• if \( m \neq n \), assume without loss of generality \( m < n \). So when we perform the same integration by parts \( n \) times we have

\[
\int_{-\infty}^{\infty} H_m(x) \left[ \frac{d^n}{dx^n} e^{-x^2} \right] dx = (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{dx^n} H_m(x) e^{-x^2} dx
\]

\[
= (-1)^n \int_{-\infty}^{\infty} 0 e^{-x^2} dx
\]

\[
= 0
\]

So in summary,

\[
(H_n, H_m) = \begin{cases} \sqrt{\pi} 2^n n! & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}
\]

and so the Hermite polynomials are orthogonal to each other.

c. Differential Equation. We wish to verify the differential equation

\[
H''_n - 2xH'_n + 2nH_n = 0.
\]

Consider that

\[
H''_n = \frac{d^2}{dx^2} \left[ (-1)^n e^{-x^2} \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right]
\]

\[
= (-1)^n \frac{d}{dx} \left[ e^{x^2} \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\} + 2xe^{x^2} \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right]
\]

\[
= (-1)^n e^{x^2} \left[ \frac{d^{n+2}}{dx^{n+2}} \left\{ e^{-x^2} \right\} + 4x \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\} + (4x^2 - 2) \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right]
\]

\[
2xH'_n = (-1)^n 2xe^{x^2} \left[ \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\} + 2x \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right]
\]

\[
2nH_n = (-1)^n 2ne^{x^2} \left[ \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right]
\]
So that means

\[ H''_n - 2xH'_n = (-1)^n x^2 \left[ \frac{d^{n+2}}{dx^{n+2}} \left\{ e^{-x^2} \right\} + 4x \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\} + \left( 4x^2 + 2 \right) \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right] \]

\[-(-1)^n x^2 \left[ \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\} + 2n \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right] \]

\[ = (-1)^n x^2 \left[ \frac{d^{n+2}}{dx^{n+2}} \left\{ e^{-x^2} \right\} + 2x \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\} + 2 \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right] \]

\[ H''_n - 2xH'_n + 2nH_n = (-1)^n x^2 \left[ \frac{d^{n+2}}{dx^{n+2}} \left\{ e^{-x^2} \right\} + 2x \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\} + 2 \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right] \]

\[ + (-1)^n 2n x e^{-x^2} \left[ \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right] \]

\[ = (-1)^n x^2 \left[ \frac{d^{n+2}}{dx^{n+2}} \left\{ e^{-x^2} \right\} + 2x \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\} + 2(n+1) \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \right] \]

\[ = (-1)^n x^2 \left[ \frac{d^{n+2}}{dx^{n+2}} \left\{ e^{-x^2} \right\} \right] \]

\[ - \frac{2x(-1)^{n+1} x e^{-x^2} \frac{d^{n+1}}{dx^{n+1}} \left\{ e^{-x^2} \right\}} \]

\[ + 2(n+1)(-1)^n x^2 \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\} \]

\[ = H_{n+2} - 2xH_{n+1} + 2(n+1)H_n \]

This is zero if and only if the Hermite 3-term recursion relation is true, which is proven below.

d. **3-Term Recursion Relation.** We wish to verify the recursion relationship

\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \]

We perform this by induction. First, given the definitions in part a,

\[ H_2 = 4x^2 - 2x = (2x)(2x) - (2 * 1)(1) = 2x * H_1 - 2 * 1 * H_0 \]
so it holds in the base case of \( n = 1 \).

Next, we assume \( H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \). This implies

\[
\begin{align*}
(-1)^{n+1}e^{-x^2} \frac{d^{n+1}}{dx^{n+1}} \{e^{-x^2}\} &= 2x(-1)^ne^{-x^2} \frac{d^n}{dx^n} \{e^{-x^2}\} - 2n(-1)^{n-1}e^{-x^2} \frac{d^{n-1}}{dx^{n-1}} \{e^{-x^2}\} \\
\frac{d^{n+1}}{dx^{n+1}} \{e^{-x^2}\} &= -2xe^{-x^2} \frac{d^n}{dx^n} \{e^{-x^2}\} - 2n \frac{d^{n-1}}{dx^{n-1}} \{e^{-x^2}\} \\
\frac{d^{n+2}}{dx^{n+2}} \{e^{-x^2}\} &= \frac{d}{dx} \left( 2x \frac{d^n}{dx^n} \{e^{-x^2}\} + 2n \frac{d^{n-1}}{dx^{n-1}} \{e^{-x^2}\} \right) \\
&= 2 \frac{d^n}{dx^n} \{e^{-x^2}\} + 2x \frac{d^{n+1}}{dx^{n+1}} \{e^{-x^2}\} + 2n \frac{d^n}{dx^n} \{e^{-x^2}\} \\
&= 2x \frac{d^{n+1}}{dx^{n+1}} \{e^{-x^2}\} + 2(n+1) \frac{d^n}{dx^n} \{e^{-x^2}\} \\
(-1)^{n+2}e^{-x^2} \frac{d^{n+2}}{dx^{n+2}} \{e^{-x^2}\} &= 2x(-1)^{n+1}e^{-x^2} \frac{d^{n+1}}{dx^{n+1}} \{e^{-x^2}\} + 2(n+1)(-1)^{n+1}e^{-x^2} \frac{d^{n+1}}{dx^{n+1}} \{e^{-x^2}\} \\
&= 2x(-1)^{n+1}e^{-x^2} \frac{d^{n+1}}{dx^{n+1}} \{e^{-x^2}\} + 2(n+1)(-1)^{n+1}e^{-x^2} \frac{d^{n+1}}{dx^{n+1}} \{e^{-x^2}\} \\
H_{n+2}(x) &= 2xH_{n+1}(x) - 2(n+1)H_n(x)
\end{align*}
\]

Thus, proof by induction: the recursion holds for all \( n \).